International macroeconomics (advanced level)

Lecture notes

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Contents

Outline

Aims of the course

The students of this course follow three different master programmes at the UAM:

- Master en **Economía Internacional,**
- Master en **Economía Cuantitativa,**
- Master en **Globalización y Políticas Públicas.**

This course aims to offer something for all three groups by discussing:

- some of the **theory and empirics** of international macroeconomics,
- **econometric applications** in international macroeconomics,
- the challenges for **macroeconomic policy in a globalizing world.**

Methodology	Use in international economics	Example
Difference equations	Exchange rate behaviour	Müller-Plantenberg (2006)
Differential equations	Hyperinflations	Cagan (1956)
	Currency crises	Flood and Garber (1984)
		Müller-Plantenberg (2010)
Intertemporal optimization	Dynamic general equilibrium	Obstfeld and Rogoff (1996)
	Current account determination	Obstfeld and Rogoff (1995)
Present value models	Current account determination	Bergin and Sheffrin (2000)
Continuous-time finance	Exchange rate behaviour	Dumas (1992)
		Hau and Rey (2006)
Vector autoregressions	Real exchange rate behaviour	Blanchard and Quah (1989)
		Clarida and Galí (1994)
Cointegration	Purchasing power parity	Enders (1988)
Error correction models	Exchange rate pass-through	Fujii (2006)
Nonlinear time series	Nonlinear adjustment towards PPP	Obstfeld and Taylor (1997)

Basic models

1 Balassa-Samuelson effect

The Balassa-Samuelson effect is a tendency for countries with higher productivity in tradables compared with nontradables to have higher price levels (Balassa, 1964, Samuelson, 1964).

1.1 Growth accounting

Often we can derive relationships between growth rates by

- first taking logs of variables,
- then differentiating the resulting logarithms with respect to time.

1.1.1 Example 1

$$\begin{aligned}
& z = xy \\
& \Rightarrow \log(z) = \log(x) + \log(y) \\
& \Rightarrow \frac{\dot{z}}{z} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} \\
& \Rightarrow \hat{z} = \hat{x} + \hat{y},
\end{aligned}$$

where the dot above a variable indicates the derivative of that variable with respect to time and the hat above a variable the (continuous) percentage change of that variable.

1.1.2 Example 2

$$\begin{aligned}
& z = \frac{x}{y} \\
& \Rightarrow \log(z) = \log(x) - \log(y) \\
& \Rightarrow \frac{\dot{z}}{z} = \frac{\dot{x}}{x} - \frac{\dot{y}}{y} \\
& \Rightarrow \hat{z} = \hat{x} - \hat{y},
\end{aligned}$$

1.1.3 Example 3

$$
\begin{aligned}
&z = x + y \\
&\Rightarrow \log(z) = \log(x + y) \\
&\Rightarrow \frac{\dot{z}}{z} = \frac{\dot{x} + \dot{y}}{x + y} = \frac{x}{z}\frac{\dot{x}}{x} + \frac{y}{z}\frac{\dot{y}}{y} \\
&\Rightarrow \hat{z} = \frac{x}{z}\hat{x} + \frac{y}{z}\hat{y}
\end{aligned}
$$

1.2 The price of non-traded goods with mobile capital

We consider an economy with traded and nontraded goods (p. 199–214, Obstfeld and Rogoff, 1996). We are interested to determine what drives the relative price of nontraded goods, P_N. (That is, P_N is the price of nontradables in terms of the price of tradables which for simplicity is normalized to unity, $p_T = 1$).

We make two important assumptions:

- Capital is mobile between sectors and between countries.
- Labour is mobile between sectors but not between countries.

There are two production functions, one for tradables and one for nontradables, both with constant returns to scale:

$$Y_T = A_T F(K_T, L_T), \tag{1}$$
$$Y_N = A_N G(K_N, L_N). \tag{2}$$

The assumption of constant returns to scale implies that we can work with the production function in intensive form (here, in per capita terms):

$$y_T := \frac{Y_T}{L_T} = \frac{A_T F(K_T, L_T)}{L_T} = A_T F\left(\frac{K_T}{L_T}, 1\right) = A_T F(k_T, 1) = A_T f(k_T), \tag{3}$$
$$y_N := \frac{Y_N}{L_N} = \frac{A_N G(K_N, L_N)}{L_N} = A_N G\left(\frac{K_N}{L_N}, 1\right) = A_N G(k_N, 1) = A_N g(k_N). \tag{4}$$

The marginal products of capital and labour in the tradables sector are therefore:

$$\text{MPK}_T := \frac{\partial A_T L_T f(k)}{\partial K_T} = A_T L_T f'(k) \frac{1}{L_T} = A_T f'(k), \tag{5}$$
$$\text{MPL}_T := \frac{\partial A_T L_T f(k)}{\partial L_T} = A_T \left[f(k) + L_T f'(k)\left(\frac{-K_T}{L_T^2}\right)\right] = A_T \left[f(k) - k f'(k)\right]. \tag{6}$$

The marginal products of capital and labour in the nontradables sector are:

$$\text{MPK}_N = A_N g'(k), \tag{7}$$
$$\text{MPL}_N = A_N \left[g(k) - k g'(k) \right]. \tag{8}$$

Suppose now that firms maximize the present value of their profits (measured in units of tradables):

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} [A_{T,s} F(K_{T,s}, L_{T,s}) - w_s L_{T,s} - (K_{T,s+1} - K_{T,s})], \tag{9}$$
$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} [P_{N,s} A_{N,s} F(K_{N,s}, L_{N,s}) - w_s L_{N,s} - (K_{N,s+1} - K_{N,s})]. \tag{10}$$

Profit maximization yields four equations with four unknowns (w, P_N, k_T, k_N):

$$\text{MPK}_T = A_T f'(k_T) = r, \tag{11}$$
$$\text{MPL}_T = A_T [f(k_T) - f'(k_T) k_T] = w, \tag{12}$$
$$\text{MPK}_N = P_N A_N g'(k_N) = r, \tag{13}$$
$$\text{MPL}_N = P_N A_N [g(k_N) - g'(k_N) k_N] = w. \tag{14}$$

By combining equations (11) and (12) as well as equations (13) and (14), we find that the per capita products in both sectors are equal to the per capita cost of the factor inputs:

$$A_T f(k_T) = rk_T + w, \tag{15}$$

$$P_N A_N g(k_N) = rk_N + w. \tag{16}$$

Equations (15) and (16) just represent Euler's theorem for the constant-returns-to-scale production function (in this case, in per capita terms). Both equations become more tractable once we take logs and differentiate with respect to time. We start with equation (15):

$$\log(A_T) + \log(f(k_T)) = \log(rk_T + w) \tag{17}$$

$$\Rightarrow \frac{\dot{A}_T}{A_T} + \frac{f'(k_T)\dot{k}_T}{f(k_T)} = \frac{r\dot{k}_T + \dot{w}}{rk_T + w} = \frac{r\dot{k}_T + \dot{w}}{A_T f(k_T)} \tag{18}$$

$$\Rightarrow \hat{A}_T + \frac{rk_T}{A_T f(k_T)}\hat{k}_T = \frac{rk_T}{A_T f(k_T)}\hat{k}_T + \frac{w}{A_T f(k_T)}\hat{w} \tag{19}$$

$$\Rightarrow \hat{A}_T = \mu_{LT}\hat{w}, \tag{20}$$

where

$$\mu_{LT} := \frac{w}{A_T f(k_T)}. \tag{21}$$

Note that we assume that the interest rate is constant. For equation (16) we get a similar result:

$$\hat{P}_N + \hat{A}_N = \mu_{LN}\hat{w}, \tag{22}$$

where

$$\mu_{LN} := \frac{w}{A_N g(k_N)}. \tag{23}$$

As seems intuitive, wages in the traded and nontraded goods sectors are determined by the productivity growth rates and wage shares in both sectors. By combining the last two results, we find that the relative price of nontradables grows according to the following equation:

$$\hat{P}_N = \frac{\mu_{LN}}{\mu_{LT}}\hat{A}_T - \hat{A}_N. \tag{24}$$

Note that it is plausible to assume that the production of nontradables is relatively labour-intensive:

$$\frac{\mu_{LN}}{\mu_{LT}} \geq 1. \tag{25}$$

1.3 Balassa-Samuelson effect

We assume there are two countries:

- Traded goods have the same price at home and abroad (equal to unity).
- Nontraded goods have distinct prices at home and abroad, P_N and P_N^*.

We suppose further that the domestic and foreign price levels are geometric averages of the prices of tradables and nontradables:

$$P = P_T^{\gamma} P_N^{1-\gamma} = P_N^{1-\gamma}, \tag{26}$$

$$P^* = (P_T^*)^{\gamma} (P_N^*)^{1-\gamma} = (P_N^*)^{1-\gamma}. \tag{27}$$

The real exchange rate thus depends only on the relative prices of nontradables:

$$Q = \frac{P}{P^*} = \left(\frac{P_N}{P_N^*}\right)^{1-\gamma} \tag{28}$$

To see how the inflation rates differ in both countries, we can log-differentiate this ratio:

$$\begin{aligned} \hat{P} - \hat{P}^* &= (1-\gamma)(\hat{P}_N - \hat{P}_N^*) \\ &= (1-\gamma)\left[\frac{\mu_{LN}}{\mu_{LT}}(\hat{A}_T - \hat{A}_T^*) - (\hat{A}_N - \hat{A}_N^*)\right] \\ &= (1-\gamma)\left[\left(\frac{\mu_{LN}}{\mu_{LT}}\hat{A}_T - \hat{A}_N\right) - \left(\frac{\mu_{LN}}{\mu_{LT}}\hat{A}_T^* - \hat{A}_N^*\right)\right] \end{aligned} \tag{29}$$

The country with the higher productivity growth in tradables compared with nontradables experiences a real appreciation over time (for example, Japan versus the United States in the second half of the twentieth century).

The reasoning here can also explain why rich countries tend to have higher price levels:

- Rich countries have become rich due to higher productivity growth.
- In general, productivity growth in rich countries has been particularly high in the tradables sector compared with nontradables sector.

1.4 Accounting for real exchange rate changes

Let us now turn to the question how the prices of nontraded goods affect the real exchange rate at different horizons.

First, we express the real exchange rate in terms of tradables and nontradables prices (all in logarithms):

$$\begin{aligned} q &= s + p - p^* \\ &= s + \gamma(p_T - p_T^*) + (1-\gamma)(p_N - p_N^*) \\ &= s + (p_T - p_T^*) + (1-\gamma)\left[(p_N - p_T) - (p_N^* - p_T^*)\right] \\ &= x + y, \end{aligned} \tag{30}$$

where

$$x = s + (p_T - p_T^*),$$
$$y = (1-\gamma)\left[(p_N - p_T) - (p_N^* - p_T^*)\right].$$

Differentiation with respect to time yields:

$$\hat{q} = \hat{x} + \hat{y} \tag{31}$$
$$= \hat{s} + (\hat{p}_T - \hat{p}_T^*) + (1-\gamma)\left[(\hat{p}_N - \hat{p}_T) - (\hat{p}_N^* - \hat{p}_T^*)\right]. \tag{32}$$

1.4.1 Theory versus empirics

- According to the Balassa-Samuelson hypothesis, most of the changes in the real exchange rate at long horizons are accounted for by differences in the relative prices of nontradable goods, y.
- Similarly, most of the recent literature on real exchange rates emphasizes movements in the nontraded-goods component, y.
- However, Engel (1999) has shown empirically that the nontraded-goods component, y, has accounted for little of the movement in real exchange rates [...] at any horizon:

> While I cannot be very confident about my findings at longer horizons, knowledge of the behaviour of the relative price on nontraded goods contributes practically nothing to one's understanding of [...] real exchange rates.

1.4.2 Real appreciation of the yen

Engel (1999) discusses whether the real appreciation of the yen over recent decades can be accounted for by changes in the relative prices of nontradables:

- Nontraded-goods prices have risen steadily relative to traded-goods prices in Japan since 1970; at the same time, the yen has, consistent with the theory, appreciated considerably in real terms.
- However, the rise in nontraded-goods prices may not be responsible for the rise of the yen after all:
  - First, the increase in the relative price of nontraded goods in Japan was about 40%, whereas the real exchange rate appreciated around 90%.
  - Second, the relative price of nontradables rose rather monotonously, yet there were periods of strong depreciation of the yen.
  - Finally, the relative price of nontradables rose elsewhere as well, reducing the size of y. For instance, the relative price of nontradables in the United States has closely mirrored the relative price of nontradables in Japan.

1.4.3 Conclusions

- At long horizons, the Balassa-Samuelson hypothesis may be valid and differences in relative prices may be responsible for movements in the real exchange rate.
- At least at short and medium horizons, however, it is the difference of tradable-goods prices that is mainly responsible for the movements of the real exchange rate.
- It is quite possible that changes in the real exchange rate stem primarily from changes in the nominal exchange rates, even at rather long horizons.

Difference equations

2 Introduction to difference equations

Much of economic analysis, particularly in macroeconomics, nowadays centers on the analysis of time series.

Time series analysis:

- Time series analysis is concerned with the estimation of difference equations containing stochastic components.

2.1 Definition

Difference equations express the value of a variable in terms of:

- its own lagged values,

- time and other variables.

2.2 Examples

2.2.1 Difference equation with trend, seasonal and irregular

$$y_t = T_t + S_t + I_t \qquad \text{observed variables,} \tag{33}$$
$$T_t = 1 + 0,1t \qquad \text{trend,} \tag{34}$$
$$S_t = 1,6 \sin\left(\frac{\pi}{6}t\right) \qquad \text{seasonal,} \tag{35}$$
$$I_t = 0,7I_{t-1} + \varepsilon_t \qquad \text{irregular.} \tag{36}$$

Equation (33) is a difference equation.

2.2.2 Random walk

Stock price modelled as random walk:

$$y_{t+1} = y_t + \varepsilon_{t+1},$$

where

$$\begin{aligned} y_t &= \text{stock price}, \\ \varepsilon_{t+1} &= \text{random disturbance}. \end{aligned}$$

Test:

$$\Delta y_{t+1} = \alpha_0 + \alpha_1 y_t + \varepsilon_{t+1}.$$

H_0: $\alpha_0 = 0,\ \alpha_1 = 0.$

H_1: otherwise.

2.2.3 Reduced-form and structural equations

Samuelson's (1939) classic model:

$$y_t = c_t + i_t, \tag{37}$$

$$c_t = \alpha y_{t-1} + \varepsilon_{c,t}, \qquad 0 < \alpha < 1, \tag{38}$$

$$i_t = \beta(c_t - c_{t-1}) + \varepsilon_{i,t}, \qquad \beta > 0, \tag{39}$$

where

$$\begin{aligned} y_t &:= \text{real GDP}, \\ c_t &:= \text{consumption}, \\ i_t &:= \text{investment}, \\ \varepsilon_{c,t} &\sim (0, \sigma_c^2), \\ \varepsilon_{i,t} &\sim (0, \sigma_i^2). \end{aligned} \tag{40}$$

Structural equation

A structural equation expresses an endogenous variable in terms of:

- the current realization of another endogenous variable (among other variables)

Reduced-form equation

A reduced-form equation is one expressing the value of a variable in terms of:

- its own lags,
- lags of other endogenous variables,
- current and past values of exogenous variables,
- disturbance terms.

Therefore,

- equation (37) is a structural equation,
- equation (38) is a reduced-form equation,
- equation (39) is a structural equation,

Equation (39) in reduced form:

$$i_t = \alpha\beta(y_{t-1} - y_{t-2}) + \beta(\varepsilon_{c,t} - \varepsilon_{c,t-1}) + \varepsilon_{i,t}. \tag{41}$$

Equation (39) in univariate reduced form:

$$y_t = \alpha(1+\beta)y_{t-1} - \alpha\beta y_{t-2} + (1+\beta)\varepsilon_{c,t} - \beta\varepsilon_{c,t-1} + \varepsilon_{i,t}. \tag{42}$$

2.2.4 Error correction

The Unbiased Forward Rate (UFR) hypothesis asserts:

$$s_{t+1} = f_t + \varepsilon_{t+1} \tag{43}$$

with

$$\mathrm{E}_t(\varepsilon_{t+1}) = 0, \tag{44}$$

where

$$f_t = \text{forward exchange rate.} \tag{45}$$

We can test the UFR hypothesis as follows:

$$s_{t+1} = \alpha_0 + \alpha_1 f_t + \varepsilon_{t+1}, \tag{46}$$

$$\begin{aligned} &\mathrm{H}_0: \quad \alpha_0 = 0, \quad \alpha_1 = 1, \quad \mathrm{E}_t(\varepsilon_{t+1}) = 0, \\ &\mathrm{H}_1: \quad \text{otherwise.} \end{aligned} \tag{47}$$

Adjustment process:

$$s_{t+2} = s_{t+1} - \alpha(s_{t+1} - f_t) + \varepsilon_{s,t+2}, \qquad \alpha > 0, \tag{48}$$

$$f_{t+1} = f_t + \beta(s_{t+1} - f_t) + \varepsilon_{f,t+1}, \qquad \beta > 0. \tag{49}$$

2.2.5 General form of difference equation

An nth-order difference equation with constant coefficients can be written as follows:

$$y_t = \alpha_0 + \sum_{i=1}^{n} \alpha_i y_{t-i} + x_t, \tag{50}$$

where x_t is a forcing process, which can be a function of:

- time,
- current and lagged values of other variables,
- stochastic disturbances.

2.2.6 Solution to a difference equation

The solution to a difference equation is a function of:

- elements of the forcing process x_t,
- time t,
- initial conditions (given elements of the y sequence).

Example:

$$y_t = y_{t-1} + 2, \qquad \text{difference equation,} \tag{51}$$

$$y_t = 2t + c, \qquad \text{solution.} \tag{52}$$

2.3 Lag operator

The lag operator L (backshift operator) is defined as follows:

$$L^i y_t = y_{t-i}, \qquad i = 0, \pm 1, \pm 2, \ldots \tag{53}$$

Some implications:

$$Lc = c, \quad \text{where } c \text{ is a constant,} \tag{54}$$

$$(L^i + L^j) y_t = L^i y_t + L^j y_t = y_{t-i} + y_{t-j}, \tag{55}$$

$$L^i L^j y_t = L^i y_{t-j} = y_{t-i-j}, \tag{56}$$

$$L^i L^j y_t = L^{i+j} y_t = y_{t-i-j}, \tag{57}$$

$$L^{-i} y_t = y_{t+i}. \tag{58}$$

2.4 Solving difference equations by iteration

2.4.1 Sums of geometric series

Note that when $|k| < 1$,

$$\sum_{i=0}^{m} k^i = \frac{1-k^{m+1}}{1-k} \quad \text{and} \quad \lim_{m\to\infty}\sum_{i=0}^{m} k^i = \frac{1}{1-k}, \tag{59}$$

since

$$1 + k + k^2 + \ldots + k^m = \frac{1-k^{m+1}}{1-k} \tag{60}$$

$$\begin{aligned}\Leftrightarrow \quad & (1-k)(1+k+k^2+\ldots+k^m) \\ & \quad = 1 - k + k - k^2 + \ldots + k^{m-1} - k^m + k^m - k^{m+1} \\ & \quad = 1 - k^{m+1}.\end{aligned} \tag{61}$$

Note that when $|k| > 1$,

$$\sum_{i=0}^{m} k^{-i} = \frac{-k+k^{-m}}{1-k} \quad \text{and} \quad \lim_{m\to\infty}\sum_{i=0}^{m} k^{-i} = \frac{-k}{1-k}, \tag{62}$$

since

$$\sum_{i=0}^{m} k^{-i} = \sum_{i=0}^{m} \left(k^{-1}\right)^{i} = \frac{1-\left(k^{-1}\right)^{m+1}}{1-\left(k^{-1}\right)} = \frac{k-k^{-m}}{k-1} = \frac{-k+k^{-m}}{1-k}. \tag{63}$$

2.4.2 Iteration with initial condition - case where $|a_1| < 1$

Consider the first-order linear difference equation:

$$y_t = a_0 + a_1 y_{t-1} + x_t. \tag{64}$$

Iterating forward, using a given initial condition:

$$\begin{aligned}
y_1 &= a_0 + a_1 y_0 + x_1 \\
y_2 &= a_0 + a_1 y_1 + x_2 \\
&= a_0 + a_1(a_0 + a_1 y_0 + x_1) + x_2 \\
&= a_0 + a_0 a_1 + a_1^2 y_0 + a_1 x_1 + x_2 \\
&\dots \\
y_t &= a_0 \sum_{i=0}^{t-1} a_1^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i x_{t-i}.
\end{aligned} \tag{65}$$

2.4.3 Iteration with initial condition - case where $|a_1| = 1$

What if $|a| = 1$?

$$y_t = a_0 + y_{t-1} + x_t \quad \Leftrightarrow \quad \Delta y_t = a_0 + x_t. \tag{66}$$

Iterate forward:

$$\begin{aligned} y_1 &= a_0 + y_0 + x_1 \\ y_2 &= a_0 + y_1 + x_2 \\ &= a_0 + a_0 + a_1 y_0 + x_1 + x_2 \\ &= a_0 + a_0 + y_0 + x_1 + x_2 \\ &\dots \\ y_t &= a_0 t + y_0 + \sum_{i=1}^{t} x_{t-i}. \end{aligned} \tag{67}$$

2.4.4 Iteration without initial condition - case where $|a_1| < 1$

Iterating backward:

$$\begin{aligned} y_t &= a_0 + a_1 y_{t-1} + x_t \\ &= a_0 + a_1(a_0 + a_1 y_{t-2} + x_{t-1}) + x_t \\ &= a_0 + a_0 a_1 + a_1^2 y_{t-2} + x_t + a_1 x_{t-1} \\ &= \dots \\ &= a_0 \sum_{i=0}^{m} a_1^i + a_1^{m+1} y_{t-m-1} + \sum_{i=0}^{m} a_1^i x_{t-i}. \end{aligned} \tag{68}$$

If $|a_1| < 1$, we therefore obtain the following solution:

$$y_t = a_0 \frac{1 - a_1^{m+1}}{1 - a_1} + a_1^{m+1} y_{t-m-1} + \sum_{i=0}^{m} a_1^i x_{t-i}, \tag{69}$$

which in the limit simplifies to:

$$y_t = \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i x_{t-i}. \tag{70}$$

A more general solution:

$$y_t = Aa_1^t + \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i x_{t-i}. \tag{71}$$

2.4.5 Iteration without initial condition - case where $|a_1| > 1$

To obtain a converging solution when $|a_1| > 1$, it is necessary to invert equation (64) and to iterate it forward:

$$y_t = a_0 + a_1 y_{t-1} + x_t \tag{72}$$

$$\begin{aligned} \Leftrightarrow \quad y_t &= -\frac{a_0}{a_1} + \frac{1}{a_1} y_{t+1} - \frac{1}{a_1} x_{t+1} \\ &= -\frac{a_0}{a_1} \sum_{i=0}^{m} \left(\frac{1}{a_1}\right)^i + \left(\frac{1}{a_1}\right)^{m+1} y_{t+m+1} - \sum_{i=0}^{m} \left(\frac{1}{a_1}\right)^{i+1} x_{t+i+1} \\ &= -\frac{a_0}{a_1} \frac{-a_1 + a_1^{-m}}{1-a_1} + \left(\frac{1}{a_1}\right)^{m+1} y_{t+m+1} - \sum_{i=0}^{m} \left(\frac{1}{a_1}\right)^{i+1} x_{t+i+1} \end{aligned} \tag{73}$$

As m approaches infinity, this ”forward-looking” solution converges (unless y_t or x_t grow very fast):

$$y_t = \frac{a_0}{1-a_1} - \sum_{i=0}^{\infty}\left(\frac{1}{a_1}\right)^{i+1} x_{t+i+1} \tag{74}$$

We may write this more compactly as follows:

$$\begin{aligned} y_t &= \tilde{a}_0 + \tilde{a}_1 y_{t+1} + \tilde{b} x_{t+1} \\ &= \tilde{a}_0 \sum_{i=0}^{m} \tilde{a}_1^i + \tilde{a}^{m+1} y_{t+m+1} + \tilde{b} \sum_{i=0}^{m} \tilde{a}_1^i x_{t+i+1} \\ &= \frac{\tilde{a}_0}{1-\tilde{a}_1} + \tilde{b} \sum_{i=0}^{\infty} \tilde{a}_1^i x_{t+i+1}, \end{aligned} \tag{75}$$

where

$$\tilde{a}_0 = -\frac{a_0}{a_1}, \qquad \tilde{a}_1 = \frac{1}{a_1}, \qquad \tilde{b} = -\frac{1}{a_1}.$$

An important drawback of iterative method is that the algebra becomes very complex in higher-order equations.

2.4.6 The exchange rate as an asset price in the monetary model

In the monetary model with flexible prices, the current exchange rate, s_t, depends on the expected future exchange rate, s_t^e. Rational expectations imply that agents' expectations coincide with realized values of the exchange rate, that is, $s_t^e = s_{t+1}$. The equation determining today's nominal exchange rate then becomes:

$$s_t = \tilde{a}_1 s_{t+1} + \tilde{b} f_t \tag{76}$$

where

$$\tilde{a}_1 = \frac{b}{1+b}, \qquad \tilde{b} = \frac{1}{1+b}, \qquad f_t = -(m_t - m_t^*) + a(y_t - y_t^*) + q_t.$$

The solution to this difference equation is:

$$s_t = \tilde{b} \sum_{i=0}^{\infty} \tilde{a}_1^i f_{t+i}. \tag{77}$$

Today's exchange rate thus depends, just like an asset price, on its current and future fundamentals.

2.5 Alternative solution methodology

Consider again the first-order linear difference equation (64):

$$y_t = a_0 + a_1 y_{t-1} + x_t. \tag{78}$$

Homogeneous part of equation (64):

$$y_t - a_1 y_{t-1} = 0. \tag{79}$$

Homogeneous solution.

A solution to equation (79) is called homogeneous solution, y_t^h.

Particular solution.

A solution to equation (64) is called particular solution, y_t^p.

General solution.

The general solution to a difference equation is defined to be a particular solution plus all homogeneous solutions:

$$y_t = y_t^h + y_t^p. \tag{80}$$

In the case of equation (64):

$$y_t^h = Aa_1^t, \tag{81}$$

where A is an arbitrary constant. Using this homogeneous solution, the homogeneous part of equation (64) is satisfied:

$$Aa_1^t - a_1 A a_1^{t-1} = 0. \tag{82}$$

We already found a particular solution to equation (64):

$$y_t^p = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i x_{t-i} \qquad \text{for } |a_1| < 1. \tag{83}$$

Therefore the general solution is:

$$\begin{aligned} y_t &= y_t^h + y_t^p \\ &= Aa_1^t + \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i x_{t-i}. \end{aligned} \tag{84}$$

When initial conditions are given, the arbitrary constant A can be eliminated.

Solution methodology:

Step 1.

Find all n homogeneous solutions.

Step 2.

Find a particular solution.

Step 3.

Obtain general solution (= sum of particular solution and linear combination of all homogeneous solutions).

Step 4.

Eliminate arbitrary constants by imposing initial conditions.

2.5.1 Example: Second-order difference equation

Consider the following second-order difference equation ($n = 2$):

$$y_t = \underbrace{0.9}_{a_1} y_{t-1} - \underbrace{0.2}_{a_2} y_{t-2} + \underbrace{3}_{a_0}. \tag{85}$$

Homogeneous part:

$$y_t - 0.9y_{t-1} + 0.2y_{t-2} = 0 \tag{86}$$

Step 1.

There are two homogeneous solutions (check!):

$$\begin{aligned} y^h_{1t} &= 0.5^t, \\ y^h_{2t} &= 0.4^t. \end{aligned} \tag{87}$$

Step 2.

There is for example the following particular solution (check!):

$$y_t^p = 10. \tag{88}$$

Step 3.

Now we form the general solution:

$$y_t = A_1 0.5^t + A_2 0.4^t + 10. \tag{89}$$

Step 4.

Suppose there are the following initial conditions:

$$y_0 = 13, \quad y_1 = 11.3 \qquad \Leftrightarrow \qquad A_1 = 1, \quad A_2 = 2. \tag{90}$$

The solution with initial conditions imposed is thus:

$$y_t = 0.5^t + 2 \times 0.4^t + 10. \tag{91}$$

Remaining problems:

- How do we find homogeneous solutions to a given difference equation?
- How do we find a particular solution to a given difference equation?

2.6 Solving second-order homogeneous difference equations

2.6.1 Roots of the general quadratic equation

A quadratic equation of the form

$$ax^2 + bx + c = 0 \tag{92}$$

has the following solution:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{d}}{2a}. \tag{93}$$

When $a = 1$, the quadratic equation becomes:

$$x^2 + bx + c = 0 \tag{94}$$

The above solution simplifies to:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-b \pm \sqrt{d}}{2}. \tag{95}$$

Note that d is called the discriminant.

2.6.2 Homogeneous solutions

Consider the homogeneous part of a second-order linear difference equation:

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = 0. \tag{96}$$

We try $y_t^h = A\alpha^t$ as a homogeneous solution:

$$A\alpha^t - a_1 A\alpha^{t-1} - a_2 A\alpha^{t-2} = 0. \tag{97}$$

Note that the choice of A is arbitrary. Now divide by $A\alpha^{t-2}$:

$$\alpha^2 - a_1\alpha - a_2 = 0. \tag{98}$$

This equation is called the characteristic equation. The roots (= solutions) of this equation are called characteristic roots.

The characteristic equation of the second-order linear difference equation has the following solutions:

$$\alpha_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2} = \frac{a_1 \pm \sqrt{d}}{2}, \tag{99}$$

where $d \; (= a_1^2 + 4a_2)$ is the discriminant.

We obtain the following solution for the homogeneous equation:

$$y_t^h = A_1\alpha_1^t + A_2\alpha_2^t. \tag{100}$$

To see why this is the solution, just substitute equation (100) into equation (96):

$$A_1\alpha_1^t + A_2\alpha_2^t - a_1\left(A_1\alpha_1^{t-1} + A_2\alpha_2^{t-1}\right) - a_2\left(A_1\alpha_1^{t-2} + A_2\alpha_2^{t-2}\right) = 0 \tag{101}$$

$$\Leftrightarrow \quad A_1\left(\alpha_1^t - a_1\alpha_1^{t-1} - a_2\alpha_1^{t-2}\right) + A_2\left(\alpha_2^t - a_1\alpha_2^{t-1} - a_2\alpha_2^{t-2}\right) = 0 \tag{102}$$

$$\Leftrightarrow \quad A_1\left(\alpha_1^2 - a_1\alpha_1^1 - a_2\right) + A_2\left(\alpha_2^2 - a_1\alpha_2^1 - a_2\right) = 0. \tag{103}$$

We call α_1 and α_2 the characteristic roots of equation (96) since they are the roots of the characteristic equation (98).

Note that it is sometimes possible to guess the roots of the characteristic equation:

$$(\alpha - \alpha_1)(\alpha - \alpha_2) = 0 \tag{104}$$

$$\Leftrightarrow \quad \alpha^2 - (\alpha_1 + \alpha_2)\alpha + \alpha_1\alpha_2 = 0. \tag{105}$$

Therefore the coefficients a_1 and a_2 are related to the characteristic roots α_1 and α_2 as follows:

$$\begin{aligned} a_1 &= \alpha_1 + \alpha_2, \\ a_2 &= -\alpha_1\alpha_2. \end{aligned} \tag{106}$$

Consider for example the following equation:

$$\alpha^2 - 0.5\alpha + 0.06 = 0. \tag{107}$$

This equation has the roots

$$\alpha_1 = 0.2 \quad \text{and} \quad \alpha_2 = 0.3, \tag{108}$$

since

$$\begin{aligned} a_1 &= 0.2 + 0.3 = 0.5, \\ a_2 &= -0.2 \times 0.3 = -0.06. \end{aligned} \tag{109}$$

Depending on the value of d, we have to distinguish three cases:

Case where $d > 0$.

- The characteristic roots in this case are:

$$\alpha_{1,2} = \frac{a_1 \pm \sqrt{d}}{2}. \tag{110}$$

- The characteristic roots are real and distinct.

- The homogeneous solution is:

$$y_t^h = A_1\alpha_1^t + A_2\alpha_2^t. \tag{111}$$

- y_t is stable if $|\alpha_1| < 1$ and $|\alpha_2| < 1$.

Case where $d = 0$.

- The characteristic roots in this case are:

$$\alpha_1 = \alpha_2 = \alpha = \frac{a_1}{2}. \tag{112}$$

- The characteristic roots are real and equal.
- The homogeneous solution is:

$$y_t^h = A_1\alpha^t + A_2 t\alpha^t. \tag{113}$$

- y_t is stable if $|\alpha| < 1$.

Case where $d < 0$.

- The characteristic roots in this case are:

$$\alpha_{1,2} = \frac{\alpha_1 \pm i\sqrt{d}}{2}. \tag{114}$$

- The characteristic roots are imaginary and distinct.
- The homogeneous solution is:

$$y_t^h = \beta_1 r^t \cos(\theta t + \beta_2) \tag{115}$$

where

$$\begin{aligned} \beta_{1,2} &= \text{arbitrary constants}, \\ r &= \sqrt{-a_2}, \\ \theta &= \arccos\left(\frac{a_1}{2r}\right). \end{aligned} \tag{116}$$

- y_t is stable if $r < 1$.

2.6.3 Particular solutions

Let us now turn to the question of how to find a particular solution to a second-order linear difference equation:

$$y_t - a_1 y_{t-1} - a_2 y_{t-2} = c_t. \tag{117}$$

In a number of important cases, there are functions that are known to work as particular solutions. Here are some examples:

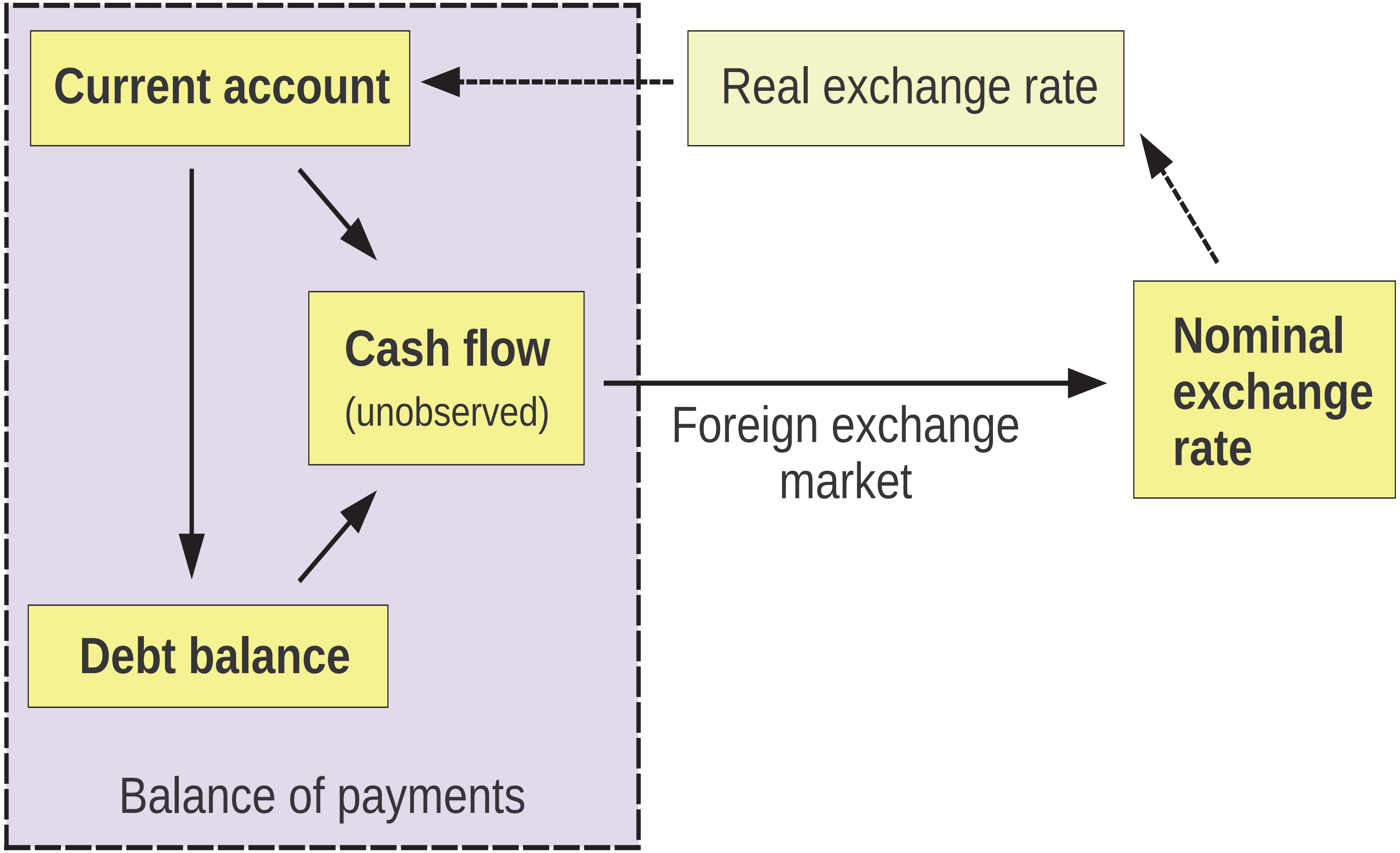
$$c_t = c \qquad y_t^p = A, \tag{118a}$$
$$c_t = ct + d \qquad y_t^p = At + B, \tag{118b}$$
$$c_t = t^n \qquad y_t^p = A_0 + A_1 t + \ldots + A_n t^n, \tag{118c}$$
$$c_t = c^t \qquad y_t^p = Ac^t, \tag{118d}$$
$$c_t = \alpha \sin(ct) + \beta \cos(ct) \qquad y_t^p = A \sin(ct) + B \cos(ct), \tag{118e}$$

The constants can be determined by the method of undetermined coefficients:

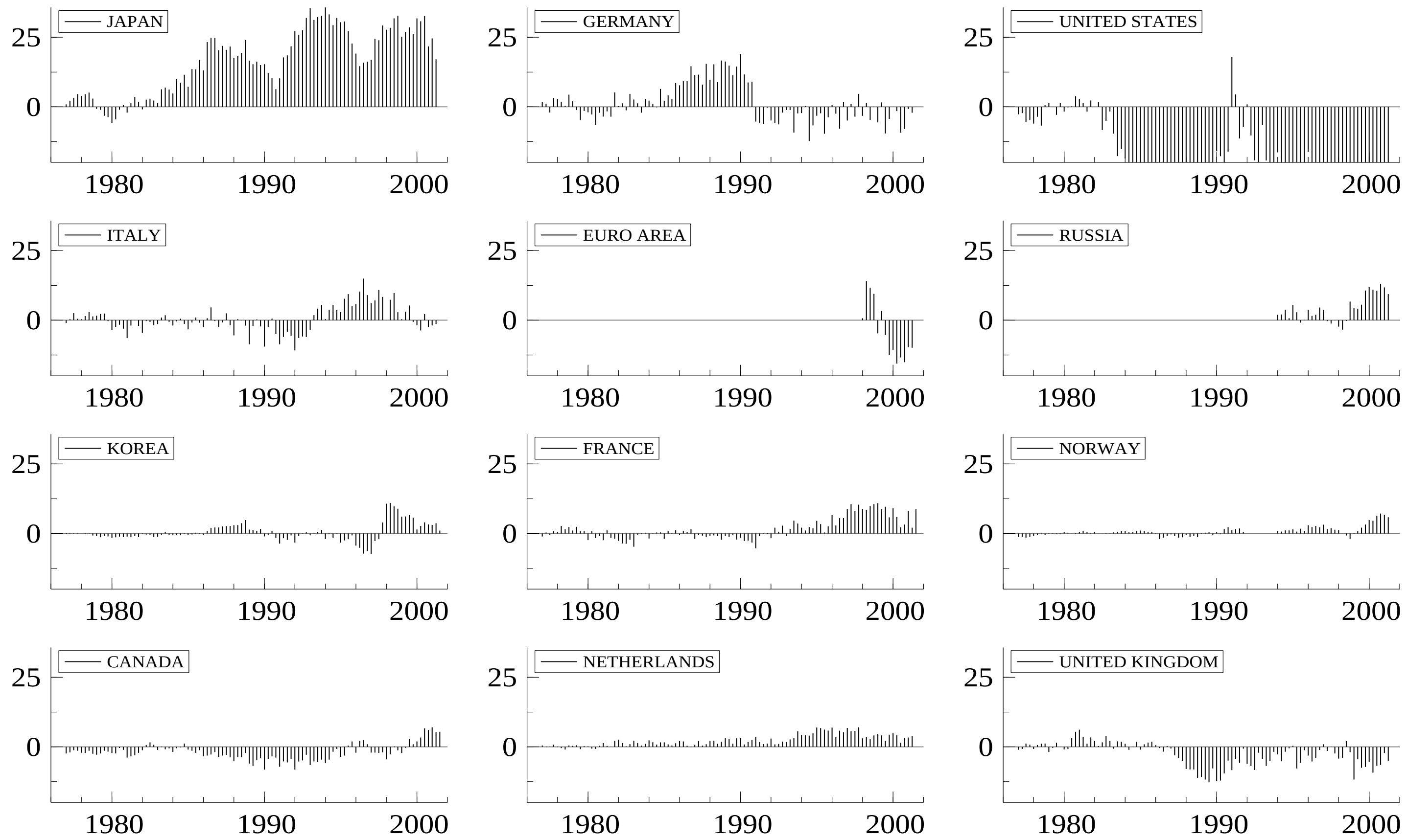
- Substitute the solution (118) into equation (117).
- Determine the constant A and B in terms of the other constants.

3 Modelling currency flows using difference equations

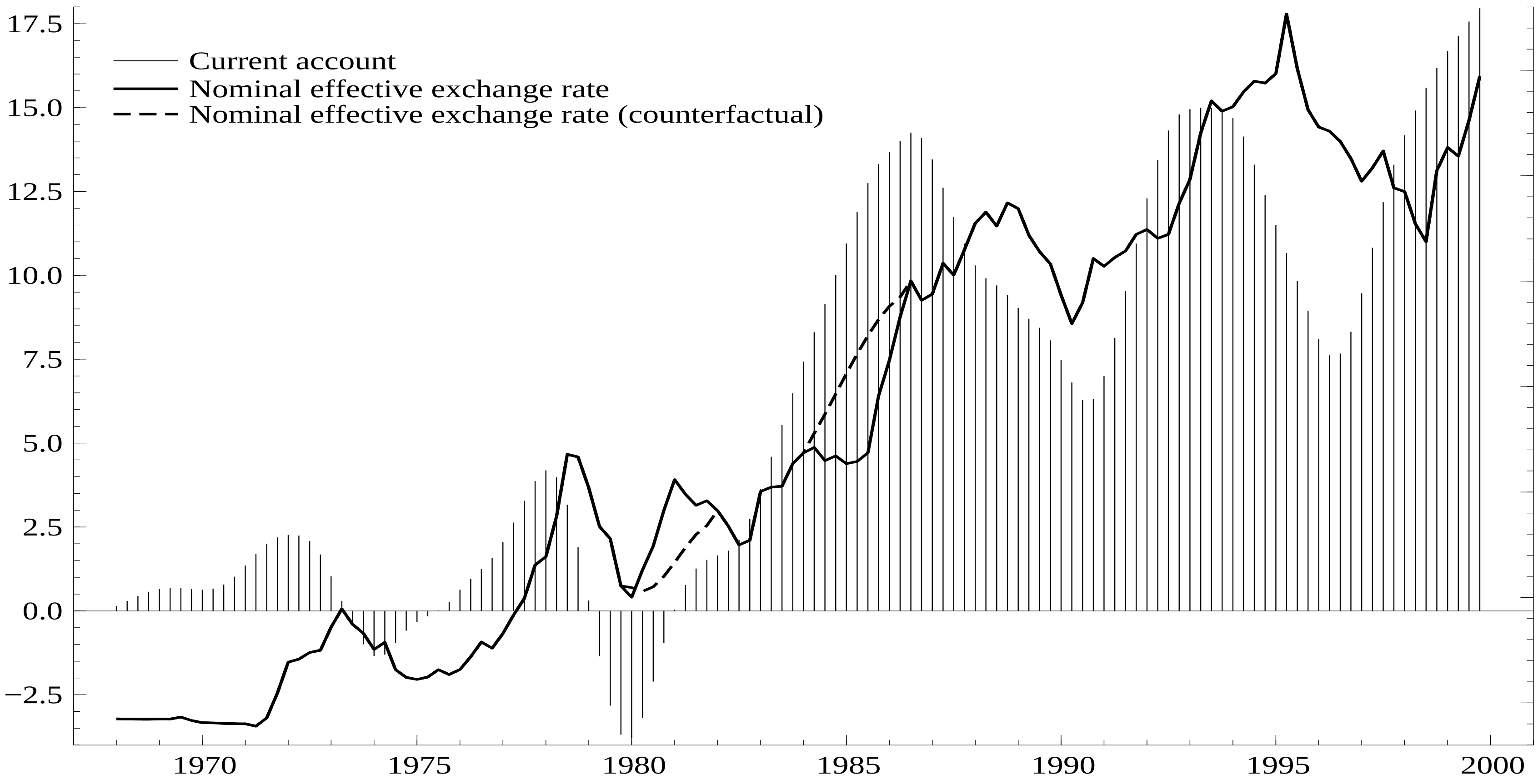
See Müller-Plantenberg (2006). The basic idea is conveyed in the diagram.



Cash flow and exchange rate determination.



Large current account surpluses.



Japanese current account and counterfactual exchange rate.

3.1 A benchmark model

The benchmark model consists of the following equations:

$$s_t = -\xi c_t, \tag{119}$$

$$q_t = s_t, \tag{120}$$

$$z_t + c_t = 0, \tag{121}$$

$$z_t = z_{t-1} - \phi q_{t-1}, \tag{122}$$

where

$q_t =$ real exchange rate,

$s_t =$ nominal exchange rate,

$z_t =$ current account,

$c_t =$ monetary account (= minus country's cash flow),

$\phi, \xi > 0.$

Whereas the parameter ϕ measures the exchange rate sensitivity of trade flows, the parameter ξ determines how the nominal exchange rate is affected by a country's international cash flow, c_t.

Transform model into first-order difference equation in the current account variable, z_t:

$$z_t = (1 - \phi\xi) z_{t-1}.$$

The solution to this equation is:

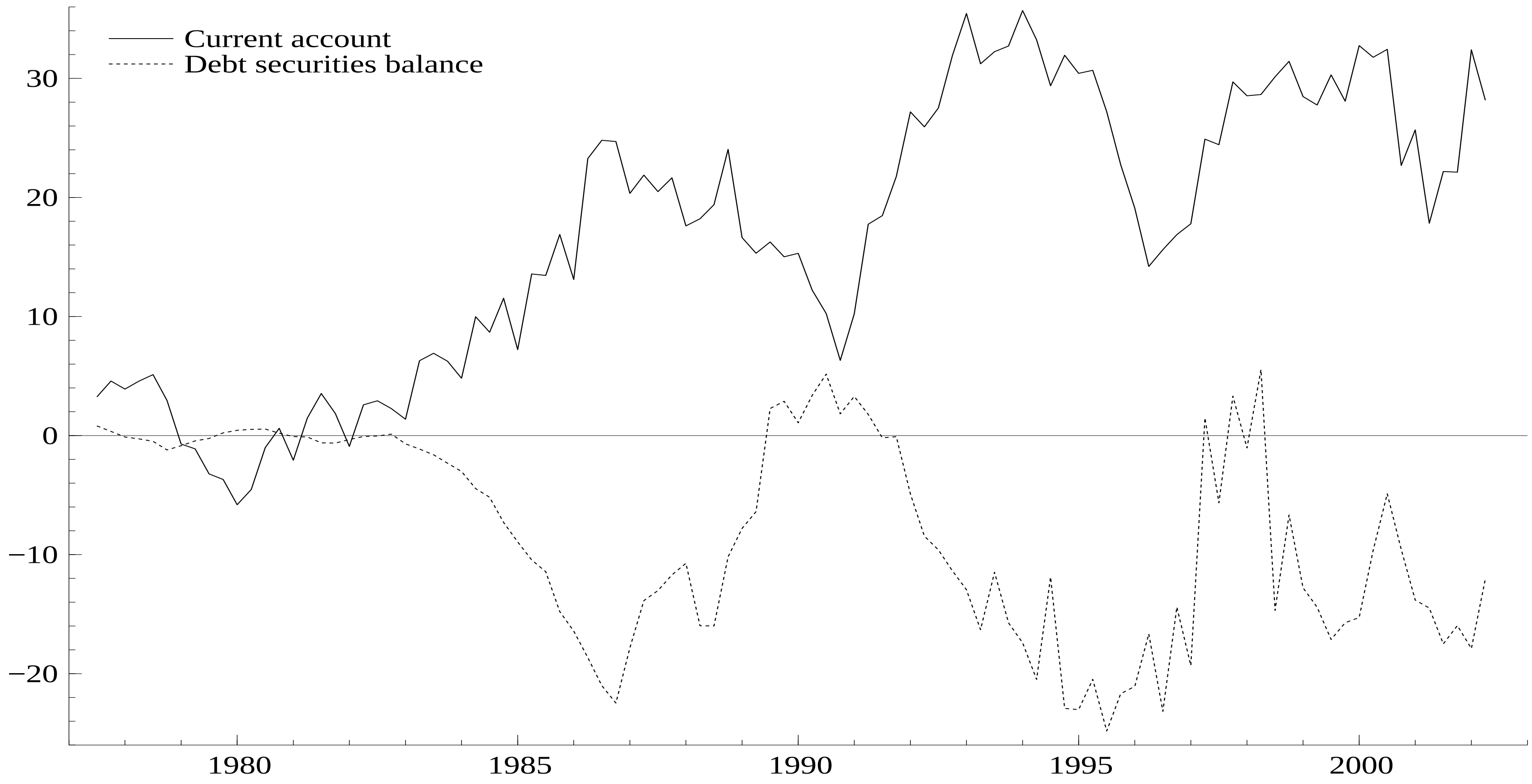
$$z_t = A(1 - \phi\xi)^t,$$

where A is an arbitrary constant.

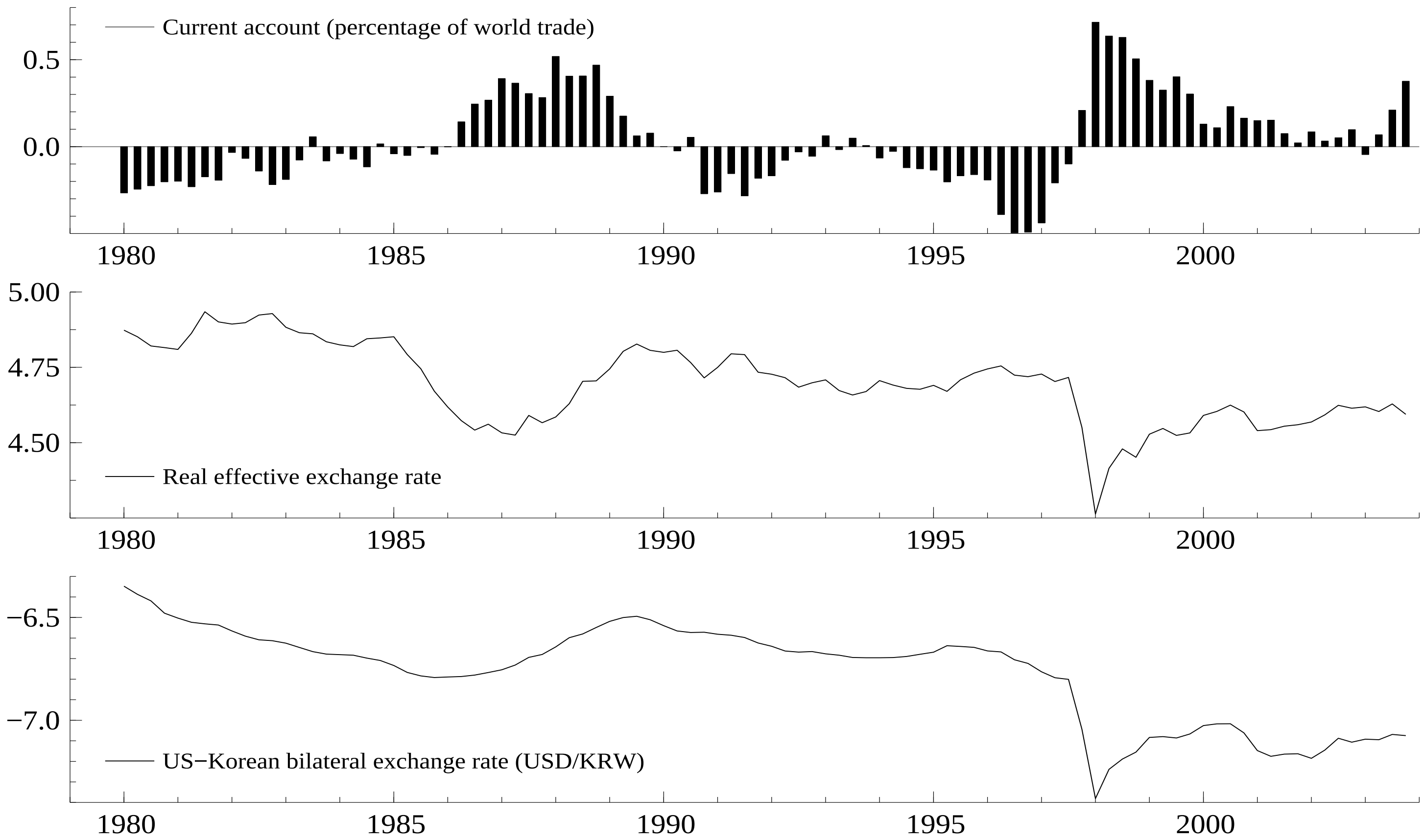
Now the solution for s_t, q_t and c_t can be derived from the model's equations.

We make the following observations:

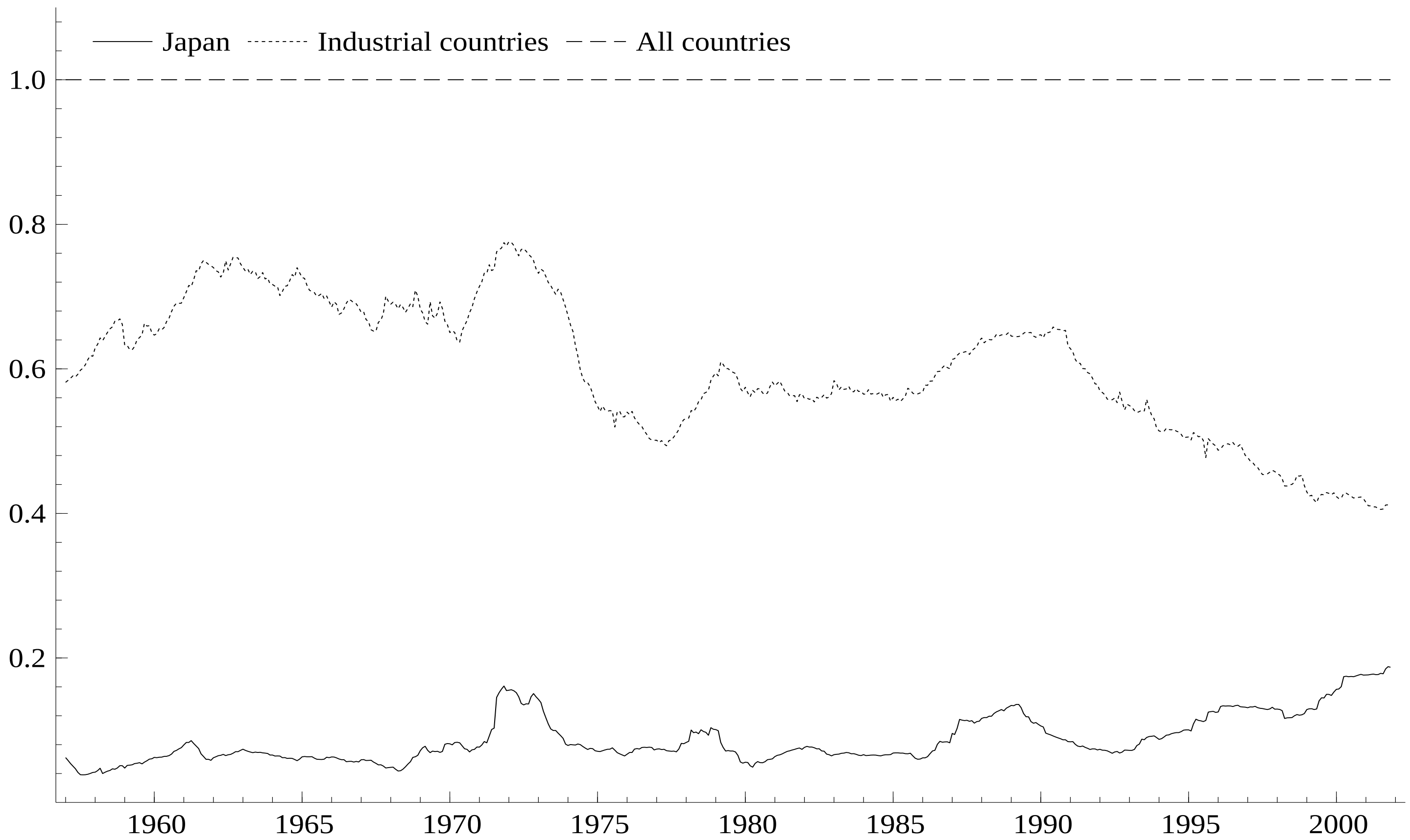
- When $\phi\xi > 1$, the current account and all the other variables in the model start to oscillate from one period to the next.
- As soon as $\phi\xi > 2$, the model's dynamic behaviour becomes explosive.
- The current account, z_t, and the real exchange rate, q_t, are positively correlated.



Current account and lending in Japan.



Korea's current account and exchange rate.



Japan's share of world reserves.

3.2 A model with international debt

We have so far assumed that countries pay for their external transactions immediately.

We shall now make the more realistic assumption that countries finance their external deficits by borrowing from abroad. Specifically, they use debt with a one-period maturity to finance their international transactions.

Another assumption we adopt is that debt flows are merely accommodating current account imbalances, that is, we exclude independently fluctuating, autonomous capital flows from our analysis.

The previous model is modified as follows:

$$s_t = -\xi c_t, \tag{123}$$

$$q_t = s_t, \tag{124}$$

$$z_t + d_t + c_t = 0, \tag{125}$$

$$d_t := d_t^1 - d_{t-1}^1, \tag{126}$$

$$c_t = d_{t-1}^1, \tag{127}$$

$$z_t = z_{t-1} - \phi q_{t-1}, \tag{128}$$

where

$$\begin{aligned} d_t &:= \text{debt balance (part of financial account of the balance of payments)}, \\ d_t^1 &:= \text{flow of foreign debt with a one-period maturity, created in period } t. \end{aligned} \tag{129}$$

Observe that equations (125), (126) and (127) imply that countries pay for their imports and receive payments for their exports always after one period:

$$c_t = -z_{t-1}. \tag{130}$$

Due to the deferred payments, adjustments now take longer than in the previous model. The model can be reduced to a second-order difference equation in the current account variable, z_t:

$$z_t = z_{t-1} - \phi\xi z_{t-2}. \tag{131}$$

As long as $\phi\xi > \frac{1}{4}$, the solution to this equation is the following trigonometric function:

$$z_t = B_1 r^t \cos(\theta t + B_2), \tag{132}$$

where

$$\begin{aligned} r &:= \sqrt{\phi\xi}, \\ \theta &:= \arccos\left(\frac{1}{2\sqrt{\phi\xi}}\right), \\ \theta &\epsilon [0, \pi]. \end{aligned} \tag{133}$$

We make the following observations:

- As in the previous model, the variables move in a cyclical fashion. However, oscillating behaviour occurs already when $\phi\xi > \frac{1}{4}$ (before, the condition was that $\phi\xi > 1$).
- Whereas the frequency of the cycles, say ω, was one-half in the previous model—the variables were oscillating from one period to the next, completing one cycle in two periods—in this model ω is strictly less than one-half.
- The present model becomes unstable as soon as $\phi\xi > 1$. In the previous model, the corresponding condition was that product of the parameters had to be greater than two, $\phi\xi > 2$. In other words, balance of payments and exchange rate fluctuations are potentially less stable when countries borrow from, and lend to, each other. With international borrowing and lending, exchange rate adjustment is slower, implying that balance of payments imbalances can grow larger.
- The correlation between the current account and the exchange rate is still positive; however, the exchange rate now lags the movements of the current account.

	Benchmark model	Model with debt
Oscillating behaviour	$\phi\xi > 1$	$\phi\xi > \frac{1}{4}$
Frequency of cycles	$\omega = \frac{1}{2}$	$\omega < \frac{1}{2}$
Explosive behaviour	$\phi\xi > 2$	$\phi\xi > 1$
Correlation between z and s	Corr$(z_t, s_t) = +1$	Corr$(z_t, s_{t+1}) = +1$
	since $z_1 = \frac{1}{\xi} s_t$	since $z_1 = \frac{1}{\xi} s_{t+1}$

Remarks:

- The period of the cycles in equation (132), p is:

$$p := \frac{2\pi}{\theta}. \tag{134}$$

- The frequency of the cycles in equation (132), ω, is:

$$\omega := \frac{1}{p} = \frac{\theta}{2\pi}. \tag{135}$$

- Since for there to be cycles in z_t, $\frac{1}{4} < \phi\xi < \infty$, we know that $0 < \theta < \pi$. From there we get the result regarding the frequency of the cycles:

$$0 < \omega < \frac{1}{2}. \tag{136}$$

Differential equations

4 Introduction to differential equations

Instead of using difference equations, it is sometimes more convenient to study economic models in continuous time using differential equations.

Definition:

- A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders.

5 First-order ordinary differential equations

We denote the first and second derivative of a variable x with respect to time t as follows:

$$\dot{x} := \frac{dx}{dt}, \qquad \ddot{x} := \frac{d^2x}{dt^2}. \tag{137}$$

What is a differential equation?

- In a differential equation, the unknown is a function, not a number.
- The equation includes one or more derivatives of the function.

The highest derivative of the function included in a differential equation is called its order.

Further, we distinguish ordinary and partial differential equations:

- An ordinary differential equation is one for which the unknown is a function of only one variable. In our case, that variable will be time.
- Partial differential equations are equations where the unknown is a function of two or more variables, and one or more of the partial derivatives of the function are included.

5.1 Deriving the solution to a differential equation

Consider the first-order differential equation:

$$\dot{x}(t) = ax(t) + b(t), \tag{138}$$

The function $b(t)$ is called ”forcing function”.

We can derive a solution as follows:

$$\dot{x}(t) - ax(t) = b(t) \tag{139}$$

$$\Leftrightarrow \quad \dot{x}(t)e^{-at} - ax(t)e^{-at} = b(t)e^{-at} \tag{140}$$

$$\Leftrightarrow \quad \frac{d}{dt}\left[x(t)e^{-at}\right] = b(t)e^{-at}. \tag{141}$$

Note that the term e^{-at} is called the ”integrating factor”. For $t_2 > t_1$, we obtain:

$$x(t_2)e^{-at_2} - x(t_1)e^{-at_1} = \int_{t_1}^{t_2} b(u)e^{-au}du \tag{142}$$

$$\Leftrightarrow \quad x(t_2) = x(t_1)e^{a(t_2-t_1)} + \int_{t_1}^{t_2} b(u)e^{-a(u-t_2)}du \tag{143}$$

$$\Leftrightarrow \quad x(t_1) = x(t_2)e^{-a(t_2-t_1)} - \int_{t_1}^{t_2} b(u)e^{-a(u-t_1)}du. \tag{144}$$

(145)

Case where $a < 0$.

In this case, as $t_1 \to -\infty$:

$$x(t_2) \to \int_{-\infty}^{t_2} b(u)e^{a(t_2-u)}du \tag{146}$$

$$\text{or} \qquad x(t) \to \int_{-\infty}^{t} b(u)e^{a(t-u)}du. \tag{147}$$

Case where $a > 0$.

In this case, as $t_2 \to \infty$:

$$x(t_1) \to -\int_{t_1}^{\infty} b(u)e^{-a(u-t_1)}du \tag{148}$$

$$\text{or} \qquad x(t) \to -\int_{t}^{\infty} b(u)e^{-a(u-t)}du. \tag{149}$$

5.2 Applications

5.2.1 Inflation

Suppose that inflation increases whenever money growth falls short of current inflation:

$$\dot{\pi}(t) = a\left(\pi(t) - \mu(t)\right), \tag{150}$$

where

$$\begin{aligned} \pi(t) &= \text{inflation}, \\ \mu(t) &= \text{money growth}, \\ a &> 0. \end{aligned} \tag{151}$$

We can solve for $\pi(t)$:

$$\dot{\pi}(t) = a\pi(t) + b(t), \tag{152}$$

where

$$b(t) = -a\mu(t). \tag{153}$$

Then current inflation is determined by future money growth:

$$\begin{aligned}\pi(t) &= -\int_t^{\infty} b(u)e^{-a(u-t)}du \\ &= a\int_t^{\infty} \mu(u)e^{-a(u-t)}du.\end{aligned} \tag{154}$$

5.2.2 Price of dividend-paying asset

Consider the following condition which equalizes the returns on an interest-bearing and a dividend-paying asset:

$$R = \frac{\pi(t)}{q(t)} + \frac{\dot{q}(t)}{q(t)} \tag{155}$$

$$\Leftrightarrow \quad \dot{q}(t) = Rq(t) - \pi(t), \tag{156}$$

where

$$\begin{aligned} R &= \text{interest rate (constant)}, \\ q(t) &= \text{price of dividend-paying asset}, \\ \pi(t) &= \text{dividend}. \end{aligned} \tag{157}$$

The condition implies that the current price of the dividend-paying asset depends on the present discounted value of all future dividends:

$$q(t) = \int_t^{\infty} \pi(u) e^{-R(u-t)} du. \tag{158}$$

5.2.3 Monetary model of exchange rate

Consider a continuous-time version of the monetary model of exchange rate determination:

$$m(t) - p(t) = ay(t) - bR(t), \tag{159}$$

$$q(t) = p(t) - p^*(t) + s(t), \tag{160}$$

$$R(t) = R^*(t) - \dot{s}(t). \tag{161}$$

The model can be rewritten in terms of an ordinary differential equation of the nominal exchange rate variable (for simplicity without the time argument):

$$\begin{aligned} \dot{s} &= R^* - R \\ &= \frac{1}{b}\left[(m - m^*) - (p - p^*) - a(y - y^*)\right] \\ &= \frac{1}{b}\left[(m - m^*) - (q - s) - a(y - y^*)\right] \\ &= \frac{1}{b}s + \frac{1}{b}\left[(m - m^*) - q - a(y - y^*)\right]. \end{aligned} \tag{162}$$

Solving this differential equation, we see that the current exchange rate is forward-looking and depends on its future economic fundamentals:

$$s(t) = \frac{1}{b}\int_t^\infty \left[-(m - m^*) + q + a(y - y^*)\right] e^{-\frac{1}{b}(u-t)} du. \tag{163}$$

6 Currency crises

6.1 Domestic credit and reserves

Balance sheet of a central bank:

Assets	Liablities
Bonds (D)	Currency in circulation
Official reserves (RS)	Bank deposits

$$
\begin{aligned}
M &= \text{Currency} + \text{Bank deposits} \\
&= RS + D \\
&= \frac{RS + D}{D} \times D \\
&= e^{\rho} D,
\end{aligned}
$$

where

$$
\begin{aligned}
RS &= \text{official reserves}, \\
D &= \text{domestic credit}, \\
\rho &= \text{index of official reserves } (\geq 0).
\end{aligned}
$$

In logarithms:

$$m = \rho + d.$$

The central bank creates money:

- by buying domestic bonds ($d \uparrow$),
- by buying foreign reserves ($\rho \uparrow$).

The monetary model can therefore be modified as follows:

$$s = -(d - d^*) - (\rho - \rho^*) + a(y - y^*) - b(R - R^*) + q.$$

- Given the levels of the other variables, an increase in the domestic credit (purchase of domestic bonds) as well as an increase in reserves (purchase of foreign currency and bonds) induce a depreciation of the domestic currency ($s \downarrow$).
- However, it is also for instance possible to neutralize a domestic credit expansion by running down foreign reserves, keeping the exchange rate constant.

The previous equation may also be written in terms of percentage changes:

$$\Delta s = -(\Delta d - \Delta d^*) - (\Delta\rho - \Delta\rho^*) + a(\Delta y - \Delta y^*) - b(\Delta R - \Delta R^*) + \Delta q,$$

where Δ is the difference operator (that is, $\Delta x = x_t - x_{t-1}$), or in terms of instantaneous percentage changes (derivatives of the logarithms with respect to time):

$$\dot{s} = -(\dot{d} - \dot{d}^*) - (\dot{\rho} - \dot{\rho}^*) + a(\dot{y} - \dot{y}^*) - b(\dot{R} - \dot{R}^*) + \dot{q}.$$

6.2 A model of currency crises

The model we discuss is a simplified version of Flood and Garber (1984). See also Mark (2001, chapter 11.1).

From the definition of the real exchange rate, it follows that the nominal exchange rate is determined as follows:

$$s(t) = -p(t) + p^*(t) + q(t). \tag{164}$$

For simplicity, we assume that $p^*(t) = 0$. Another assumption, which we will relax later on however, is that purchasing power parity holds so that $q(t) = 0$.

The money market is given by the following equation:

$$m(t) - p(t) = ay(t) - bR(t), \tag{165}$$

where national income, $y(t)$, is set to zero for simplicity.

Finally, we assume that uncovered interest parity holds:

$$R(t) = R(t)^* - \dot{s}(t). \tag{166}$$

We assume that $R^*(t) = 0$, again to make things simple.

To sum up, the model consists of three simplified equations:

$$s(t) = -p(t), \tag{167}$$

$$m(t) - p(t) = -bR(t), \tag{168}$$

$$R(t) = -\dot{s}(t). \tag{169}$$

In addition, we assume that the domestic credit component of the national money supply grows at rate μ:

$$m(t) = \rho(t) + d(t), \tag{170}$$

$$d(t) = d(0) + \mu t. \tag{171}$$

6.2.1 Exchange rate dynamics before and after the crisis

Using the first three equations of the model, we can derive a first-order differential equation in $s(t)$:

$$b\dot{s}(t) = s(t) + m(t) \tag{172}$$

$$\Leftrightarrow \quad \dot{s}(t) = \frac{1}{b}s(t) + \frac{1}{b}m(t). \tag{173}$$

The solution to this differential equation is:

$$s(t) = -\int_t^\infty \frac{1}{b}m(t)e^{-\frac{1}{b}(u-t)}du. \tag{174}$$

This integral may be further simplified using integration by parts. Note that integration by parts is based on the following equation:

$$\int_a^b f(x)g'(x)dx = \Big|_a^b f(x)g(x) - \int_a^b f'(x)g(x)dx. \tag{175}$$

In the case where $f(x) = x$ and $g'(x) = e^x$ for instance, which is similar to ours, we obtain:

$$\int_a^b xe^x dx = \Big|_a^b xe^x - \int_a^b e^x dx. \tag{176}$$

As regards equation (174), we have to distinguish two cases:

- the time before the attack when $m(t) = m(0) = d(0) + \rho(0)$,
- the time after the attack when $m(t) = d(0) + \mu t$.

In the first case, the exchange rate is constant:

$$\bar{s}(t) = -\int_t^{\infty} \frac{1}{b} m(0) e^{-\frac{1}{b}(u-t)} du. \tag{177}$$

$$f(u) = -\frac{1}{b} m(0), \qquad g'(u) = e^{-\frac{1}{b}(u-t)}, \tag{178}$$

$$f'(u) = 0, \qquad g(u) = -be^{-\frac{1}{b}(u-t)}. \tag{179}$$

$$\begin{aligned} \bar{s}(t) &= \Big|_t^{\infty} -\frac{1}{b} m(0) \times \left(-be^{-\frac{1}{b}(u-t)}\right) \\ &= -m(0) \\ &= -d(0) - \rho(0). \end{aligned} \tag{180}$$

This is, of course, the expected result from equation (172) when the exchange rate is fixed.

In the second case, after the exchange rate has started floating, the constant expansion of the domestic credit leads to a continued depreciation:

$$\tilde{s}(t) = -\int_t^\infty \frac{1}{b}(d(0) + \mu u)e^{-\frac{1}{b}(u-t)}du. \tag{181}$$

$$f(u) = -\frac{1}{b}(d(0) + \mu u), \qquad g'(u) = e^{-\frac{1}{b}(u-t)}, \tag{182}$$

$$f'(u) = -\frac{1}{b}\mu, \qquad g(u) = -be^{-\frac{1}{b}(u-t)}. \tag{183}$$

$$\begin{aligned}\tilde{s}(t) &= \Big|_t^\infty -\frac{1}{b}(d(0) + \mu u) \times \left(-be^{-\frac{1}{b}(u-t)}\right) - \int_t^\infty -\frac{1}{b}\mu \times \left(-be^{-\frac{1}{b}(u-t)}\right) du \\ &= -d(0) - \mu t - \mu b.\end{aligned} \tag{184}$$

6.2.2 Exhaustion of reserves in the absence of an attack

Time evolution of reserves:

$$\begin{aligned}\rho(t) &= m(t) - d(t) \\ &= m(0) - (d(0) + \mu t) \\ &= \rho(0) - \mu t.\end{aligned} \tag{185}$$

Time of exhaustion of reserves:

$$\rho(0) - \mu t_T = 0 \tag{186}$$

$$\Leftrightarrow \qquad t_T = \frac{1}{\mu}\rho(0). \tag{187}$$

6.2.3 Anticipated speculative attack

Time of speculative attack:

$$\bar{s}(t_A) = \tilde{s}(t_A) \tag{188}$$

$$\Leftrightarrow \qquad -d(0) - \rho(0) = -d(0) - \mu t_A - \mu b \tag{189}$$

$$\Leftrightarrow \qquad t_A = \frac{1}{\mu}\rho(0) - b = t_T - b. \tag{190}$$

Reserves at the time of the speculative attack:

$$\rho(t_A) = \rho(0) - \mu t_A = \rho(0) - \mu\left(\frac{1}{\mu}\rho(0) - b\right) = \mu b > 0. \tag{191}$$

Intuition:

- At the time of the attack, t_A, people change abruptly their expectations regarding the depreciation of the exchange rate:

$$\dot{s}(t) = 0 \quad \rightarrow \quad \dot{s}(t) < 0. \tag{192}$$

- Uncovered interest parity implies a discrete rise in the interest rate and thus an immediate fall of the money demand:

$$R \uparrow . \tag{193}$$

- A sudden rise in prices ($p(t) \uparrow$) would help to restore equilibrium in the money market but would imply a discrete downward jump of the exchange rate ($s(t) \downarrow$), which is not possible since speculators could make a riskless profit by selling the currency an instant before and buying it an instant after the attack.

- The sudden fall in the money demand therefore has to be neutralized by a discrete reduction of the nominal money supply, $m(t)$; that is, the central bank is forced to sell its remaining reserves in one final transaction:

$$\rho(t) \downarrow, \quad m(t) \downarrow . \tag{194}$$

6.2.4 Fundamental causes of currency crises

In the model, we can distinguish between the short-term and the long-term causes of a currency crisis:

- In the short term, a speculative attack on the domestic currency occurs because of the sudden change in exchange rate expectations which force the central bank to sell all its remaining reserves at once.
- The long-term cause of the crisis lies in the continuous expansion of the domestic credit, $d(t)$, which oblige the central bank to run down its reserves to keep the money supply constant.

However, whereas the short-term cause of the speculative attack is a central feature of the model, the long-term cause is not; domestic credit expansion merely represents an example of how a currency crisis can come about in the long run.

To see why, let us look once more at how changes in the nominal exchange rate come about (leaving aside the time argument of the functions for simplicity):

$$\begin{aligned} \dot{s} &= -\dot{p} + \dot{p}^* + \dot{q} \\ &= -(\dot{m} - \dot{m}^*) + a(\dot{y} - \dot{y}^*) - b(\dot{R} - \dot{R}^*) - c + \dot{\bar{q}} \\ &= -(\dot{\rho} - \dot{\rho}^*) - (\dot{d} - \dot{d}^*) + a(\dot{y} - \dot{y}^*) - b(\dot{R} - \dot{R}^*) + z + k + r + \dot{\bar{q}}, \end{aligned} \tag{195}$$

where

$$\begin{aligned} c &= \text{payments ("cash flow") balance} \\ &\quad \text{(determining demand and supply in foreign exchange market),} \\ z &= \text{current account,} \\ k &= \text{capital flow balance,} \\ r &= \text{changes in official reserves,} \\ \bar{q} &= \text{residual exchange rate determinants} \\ &\quad \text{(neither value nor demand differences).} \end{aligned} \tag{196}$$

- Note that we have made use here of the balance of payments identity, $z(t) + k(t) + c(t) + r(t) = 0$.
- Remember also that acquisitions of foreign assets enter the financial account of the balance of payments as debit items with a negative sign; for instance, all of the following transactions take a negative sign:

- the acquisition of foreign capital by domestic residents and the sale of domestic capital by foreigners ($k(t) < 0$, "capital outflows"),
- money inflows ($c < 0$) and
- purchases of foreign reserves by the central bank ($r < 0$).

In practice, there are two important long-term causes of currency crises:

Domestic credit expansion

- Continued domestic credit expansion ($\dot{d}(t) > 0$) leads to an increase in the domestic money supply.
- To avoid excessive growth of the money supply, the central bank must sell reserves ($\dot{\rho}(t) < 0, r(t) > 0$).
- Ultimately, the selling of foreign reserves will result in a speculative attack and a collapse of the exchange rate.
- The country could avoid a currency crisis by limiting the growth of its domestic credit.

Money outflows

- A persistent current account deficit or continued capital outflows ($z(t) < 0, k(t) < 0$) lead to large payments to foreigners ($c(t) > 0$), which drive up the demand for foreign currencies at the expense of the domestic currency.
- To stabilize the exchange rate, the central bank needs to sell its reserves ($\dot{\rho}(t) < 0, r(t) > 0$).
- Ultimately, the selling of foreign reserves will result in a speculative attack and a collapse of the exchange rate.
- Note that in this case, the depletion of reserves is not caused by growing domestic credit. Reducing domestic credit ($\dot{d}(t) < 0$) will not be a useful remedy to avoid a currency crisis since it is likely to produce a recession. (This is a lesson that was learned during the currency crises of the 1990s, particularly the Asian crisis of 1997–1998.)
- Instead it is important to stabilize the current account (for instance through a controlled depreciation, a so-called crawling peg) and to restrict capital outflows (for instance through capital controls).

7 Systems of differential equations

7.1 Uncoupling of differential equations

Consider the system of differential equations:

$$\underset{n\times 1}{\dot{\mathbf{x}}(t)} = \underset{n\times n}{\mathbf{A}} \underset{n\times 1}{\mathbf{x}(t)} + \underset{n\times 1}{\mathbf{b}(t)} . \tag{197}$$

Note that the system contains n interdependent equations so that our previous method of analysing differential equations does not apply.

However, suppose that $\mathbf{A}$ is diagonalizable, that is:

$$\mathbf{A} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1} \tag{198}$$

where

$$\begin{aligned} \mathbf{P} &= (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)_{n\times n}, \\ \boldsymbol{\Lambda} &= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)_{n\times n}, \\ \mathbf{p}_i &= i\text{th eigenvector of } \mathbf{A}, \\ \lambda_i &= i\text{th eigenvalue of } \mathbf{A}. \end{aligned} \tag{199}$$

We may now transform the original system of differential equations in (197) into a set of n independent (orthogonal) equations as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \tag{200}$$

$$\Leftrightarrow \quad \dot{\mathbf{x}}(t) = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}\mathbf{x}(t) + \mathbf{b}(t) \tag{201}$$

$$\Leftrightarrow \quad \mathbf{P}^{-1}\dot{\mathbf{x}}(t) = \boldsymbol{\Lambda}\mathbf{P}^{-1}\mathbf{x}(t) + \mathbf{P}^{-1}\mathbf{b}(t) \tag{202}$$

$$\Leftrightarrow \quad \dot{\mathbf{x}}^{*}(t) = \boldsymbol{\Lambda}\mathbf{x}^{*}(t) + \mathbf{b}^{*}(t) \tag{203}$$

Our previous method of solving differential equations may now be applied to each of the n independent equations. At any time, $\mathbf{x}(t)$ and $\mathbf{b}(t)$ may be recovered as follows:

$$\mathbf{x}(t) = \mathbf{P}\mathbf{x}^{*}(t), \qquad \mathbf{b}(t) = \mathbf{P}\mathbf{b}^{*}(t). \tag{204}$$

7.2 Dornbusch model

The Dornbusch model is presented in many textbooks, for example in Heijdra and van der Ploeg (2002) and Obstfeld and Rogoff (1996).

7.2.1 The model's equations

The Dornbusch model is based on the following relations:

$$y = -cR + dG - e(s + p - p^*), \tag{205}$$
$$m - p = ay - bR, \tag{206}$$
$$R = R^* - \dot{s}, \tag{207}$$
$$\dot{p} = f(y - \bar{y}). \tag{208}$$

- Endogenous variables: y, R, s, p.
- Exogenous variables: m, G, $\bar{y}$, p^*, R^*.
- Parameters (all positive): a, b, c, d, e, f.

7.2.2 Long-run characteristics

We may derive the long-run characteristics by setting $\dot{s} = 0$ and $\dot{p} = 0$:

- Monetary neutrality: $p = m$ in the long run, and no effect of m on y or R.
- Unique equilibrium real exchange rate:

$$
\begin{aligned}
q &= s + p - p^* \\
&= \frac{1}{e}\left(-\bar{y} - cR^* + dG\right).
\end{aligned} \tag{209}
$$

Note that the equilibrium real exchange rate is not affected by monetary policy but that it can be affected by fiscal policy.

7.2.3 Short-run dynamics

To study the short-run dynamics implied by the model, let us reduce the model to a system of two differential equations in s and p. Note first that for given values of the nominal exchange rate and the domestic price level, the domestic output and interest rate can be written as:

$$\begin{aligned} y &= \frac{c(m-p) + bdG - be(s+p-p^*)}{b+ac}, \\ R &= \frac{-(m-p) + adG - ae(s+p-p^*)}{b+ac}. \end{aligned} \tag{210}$$

$$\begin{aligned} \dot{s} &= R^* - R \\ &= R^* + \frac{(m-p) - adG + ae(s+p-p^*)}{b+ac}, \\ \dot{p} &= f(y-\bar{y}) \\ &= f\frac{c(m-p) + bdG - be(s+p-p^*)}{b+ac} - f\bar{y}. \end{aligned} \tag{211}$$

$$\begin{bmatrix} \dot{s} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{ae}{b+ac} & \frac{ae-1}{b+ac} \\ -\frac{bef}{b+ac} & -\frac{cf+bef}{b+ac} \end{bmatrix} \begin{bmatrix} s \\ p \end{bmatrix} + \begin{bmatrix} \frac{1}{b+ac} & \frac{-ad}{b+ac} & 0 & \frac{-ae}{b+ac} & 1 \\ \frac{cf}{b+ac} & \frac{bdf}{b+ac} & -f & \frac{bef}{b+ac} & 0 \end{bmatrix} \begin{bmatrix} m \\ G \\ \bar{y} \\ p^* \\ R^* \end{bmatrix}. \tag{212}$$

We shall assume that $ae < 1$.

In a diagram with p on the horizontal and s on the vertical axis, the $\dot{s} = 0$ curve is upward-sloping since $\dot{s} = 0$ implies:

$$s = \frac{1 - ae}{ae}p + -\frac{1}{ae}m + \frac{d}{e}G + p^* - \frac{b + ac}{ae}R^*. \tag{213}$$

On the other hand, the $\dot{p} = 0$ curve is downward-sloping since $\dot{p} = 0$ implies:

$$s = -\frac{c + be}{be}p + \frac{c}{be}m + \frac{d}{e}G - \frac{b + ac}{be}\bar{y} + p^*. \tag{214}$$

We may now analyse the model in a phase diagram with s on the vertical and p on the horizontal axis.

In doing so, we assume that:

- the exchange rate, s, is a jump variable that moves instantaneously towards any level required to achieve equilibrium in the long run and that
- the price level, p, is a crawl variable that moves continuously without abrupt jumps.

We are interested to answer the following questions:

- Is the system saddle-path stable?

  (The condition for the model to be saddle-path stable is that $|\mathbf{A}| < 0$ and is here fulfilled.)
- How is the adjustment towards the equilibrium?
- How does the equilibrium and the adjustment towards the equilibrium change if there is a change in one or several of the exogenous variables.

8 Laplace transforms

The main purpose of Laplace transforms is the solution of differential equations and systems of such equations, as well as corresponding initial value problems.

Useful introductions to Laplace transforms can be found in **?** and Kreyszig (1999).

8.1 Definition of Laplace transforms

The Laplace $F(s) = L\{f(t)\}$ of a function $f(t)$ is defined by:

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt. \tag{215}$$

It is important to note that the original function f depends on t and that its transform, the new function F, depends on s.

The original function $f(t)$ is called the inverse transform, or inverse, of $F(s)$ and we write:

$$f(t) = L^{-1}(F). \tag{216}$$

To avoid confusion, it is useful to denote original functions by lowercase letters and their transforms by the same letters in capitals:

$$f(t) \to F(s), \quad g(t) \to G(s), \quad \text{etc.} \tag{217}$$

8.2 Standard Laplace transforms

f(t)	F(s) = L{f(t)}
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2!}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{1}{s^n}$
e^{at}	$\frac{1}{s-a}$

f(t)	F(s) = L{f(t)}
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$u(t-c)$	$\frac{e^{-cs}}{s}$
$\delta(t-a)$	e^{-as}

Of course, these tables can also be used to find inverse transforms.

8.3 Properties of Laplace transforms

8.3.1 Linearity of the Laplace transform

The Laplace transform is a linear transform:

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}. \tag{218}$$

8.3.2 First shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then:

$$L\{e^{-at}f(t)\} = F(s + a). \tag{219}$$

8.3.3 Multiplying and dividing by t

If $L\{f(t)\} = F(s)$, then

$$L\{tf(t)\} = -F'(s). \tag{220}$$

If $L\{f(t)\} = F(s)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma, \tag{221}$$

provided $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists.

8.3.4 Laplace transforms of the derivatives of $f(t)$

The Laplace transforms of the derivatives of $f(t)$ are as follows:

$$\begin{aligned} L\{f'(t)\} &= sL\{f(t)\} - f(0), \\ L\{f''(t)\} &= s^2L\{f(t)\} - sf(0) - f'(0), \\ L\{f'''(t)\} &= s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0). \end{aligned} \tag{222}$$

It is convenient to adopt a more compact notation here, letting $x := f(t)$ and $\bar{x} := L\{x\}$:

$$\begin{aligned} L\{x\} &= \bar{x}, \\ L\{\dot{x}\} &= s\bar{x} - x(0), \\ L\{\ddot{x}\} &= s^2\bar{x} - sx(0) - \dot{x}(0), \\ L\{\dddot{x}\} &= s^3\bar{x} - s^2x(0) - s\dot{x}(0) - \ddot{x}(0), \\ L\{\ddddot{x}\} &= s^4\bar{x} - s^3x(0) - s^2\dot{x}(0) - s\ddot{x}(0) - \dddot{x}(0). \end{aligned} \tag{223}$$

8.3.5 Second shift theorem

The second shift theorem states that if $L\{f(t)\} = F(s)$ then:

$$L\{u(t-c)f(t-c)\} = e^{-cs}F(s), \tag{224}$$
$$L^{-1}\{e^{-cs}F(s)\} = u(t-c)f(t-c). \tag{225}$$

This theorem turns out to be useful in finding inverse transforms.

8.4 Solution of differential equations

8.4.1 Solving differential equations using Laplace transforms

Many differential equations can be solved using Laplace transforms as follows:

- Rewrite the differential equation in terms of Laplace transforms.
- Insert the given initial conditions.
- Rearrange the equation algebraically to give the transform of the solution.
- Express the transform in standard form by partial fractions.
- Determine the inverse transforms to obtain the particular solution.

8.4.2 First-order differential equations

First-order differential equation:

$$\dot{x}(t) = 2x(t) = 4, \tag{226}$$

where

$$x(0) = 1. \tag{227}$$

Solution:

- Laplace transforms:

$$(s\bar{x} - x(0)) - 2\bar{x} = \frac{4}{s}. \tag{228}$$

- Initial condition:

$$s\bar{x} - 1 - 2\bar{x} = \frac{4}{s}. \tag{229}$$

- Solve for $\bar{x}$:

$$\bar{x} = \frac{s+4}{s(s-2)}. \tag{230}$$

- Partial fractions:

$$\bar{x} = \frac{3}{s-2} - \frac{2}{s}. \tag{231}$$

- Inverse transforms:

$$x(t) = 3e^{2t} - 2. \tag{232}$$

8.4.3 Second-order differential equations

Second-order differential equation:

$$\ddot{x}(t) - 3\dot{x}(t) + 2x(t) = 2e^{3t}, \tag{233}$$

where

$$\begin{aligned} x(0) &= 5, \\ \dot{x}(0) &= 7. \end{aligned} \tag{234}$$

Solution:

- Laplace transforms:

$$\big(s^2\bar{x} - sx(0) - \dot{x}(0)\big) - 3(s\bar{x} - x(0)) + 2\bar{x} = \frac{2}{s-3}. \tag{235}$$

- Initial conditions:

$$\left(s^2\bar{x} - 5s - 7\right) - 3(s\bar{x} - 5) + 2\bar{x} = \frac{2}{s-3}. \tag{236}$$

- Solve for $\bar{x}$:

$$\bar{x} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} \tag{237}$$

- Partial fractions:

$$\bar{x} = \frac{4}{s-1} + \frac{1}{s-3}. \tag{238}$$

- Inverse transforms:

$$x(t) = 4e^t + e^{3t}. \tag{239}$$

8.4.4 Systems of differential equations

Systems of differential equations:

$$\dot{y}(t) - x(t) = e^t, \tag{240}$$

$$\dot{x}(t) + y(t) = e^{-t}, \tag{241}$$

where

$$x(0) = y(0) = 0. \tag{242}$$

Solution:

- Laplace transforms:

$$(s\bar{y} - y(0)) - \bar{x} = \frac{1}{s-1}, \tag{243}$$

$$(s\bar{x} - x(0)) + \bar{y} = \frac{1}{s+1}. \tag{244}$$

- Initial conditions:

$$s\bar{y} - \bar{x} = \frac{1}{s-1}, \tag{245}$$

$$s\bar{x} + \bar{y} = \frac{1}{s+1}. \tag{246}$$

- Solve for $\bar{x}$:

$$\bar{x} = \frac{s^2 - 2s - 1}{(s-1)(s+1)(s^2+1)}. \tag{247}$$

- Partial fractions:

$$\bar{x} = \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{s+1}{s^2+1}. \tag{248}$$

- Inverse transforms:

$$x(t) = -\frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + \cos t + \sin t. \tag{249}$$

- Obtain $y(t)$ from $y(t) = -\dot{x}(t) + e^{-t}$:

$$y(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - \cos t + \sin t. \tag{250}$$

9 Solving the model of section 6.2 using Laplace transforms

Let us use the Laplace transform method to solve once again the currency crisis model of section 6.2.

Recall the differential equation (172), which describes the nominal exchange rate's dynamics before and after the attack:

$$\dot{s}(t) = \frac{1}{b}s(t) + \frac{1}{b}m(t). \tag{251}$$

Let us consider the case where the exchange rate has already started to float after being attacked, so that $m(t) = d(0) + \mu t$. To avoid confusion with the parameter s of the Laplace transform, we use the function $x(t)$ rather than $s(t)$ to denote the exchange rate. Then we have:

$$\dot{x}(t) = \frac{1}{b}x(t) + \frac{1}{b}(d(0) + \mu t). \tag{252}$$

Here is how we can solve the differential equation for $s(t)$ using Laplace transforms:

$$s\bar{x} - x(0) = \frac{1}{b}\bar{x} + \frac{1}{b}\left(\frac{d(0)}{s} + \frac{\mu}{s^2}\right) \tag{253}$$

$$\Leftrightarrow \quad \bar{x} = \frac{bx(0)s^2 + d(0)s + \mu}{s^2(sb-1)}. \tag{254}$$

Now we take partial fractions:

$$\bar{x} = \frac{-d(0) - \mu b}{s} - \frac{\mu}{s^2} + \frac{x(0) + d(0) + \mu b}{s - \frac{1}{b}}. \tag{255}$$

Taking the inverse Laplace transforms of the resulting terms, we obtain:

$$x(t) = -d(0) - \mu b - \mu t + (x(0) + d(0) + \mu b)\, e^{\frac{1}{b}t}. \tag{256}$$

We see that there are infinitely many solutions, depending on the choice of the initial condition.

- If we choose $x(0) = -d(0) - \mu b$ as the initial condition, we obtain the linear solution already encountered in equation (184).
- However, with any other initial condition, the exchange rate will diverge exponentially from the linear trend given by $-d(0) - \mu b - \mu t$.

10 A model of currency flows in continuous time

10.1 The model's equations

We now consider model of currency flows and exchange rate movements in continuous time. The model consists of the following equations:

$$\dot{s}(t) = -\xi c(t), \tag{257}$$
$$q(t) = s(t), \tag{258}$$
$$\dot{z}(t) = -\phi q(t), \tag{259}$$
$$c(t) = -z(t). \tag{260}$$

10.2 Solving the model as a system of differential equations

Let us write the model a little more compactly:

$$\dot{q}(t) = \xi z(t), \tag{261}$$
$$\dot{z}(t) = -\phi q(t). \tag{262}$$

Solution using Laplace transforms:

- Laplace transforms:

$$\begin{aligned}(s\bar{q} - q(0)) &= \xi\bar{z}, \\ (s\bar{z} - z(0)) &= -\phi\bar{q}.\end{aligned} \tag{263}$$

- Solve for $\bar{z}$:

$$\bar{z} = \frac{sz(0) - \phi q(0)}{s^2 + \xi\phi}. \tag{264}$$

- Inverse transforms:

$$z(t) = z(0)\cos\left(\sqrt{\xi\phi}\,t\right) - \frac{\phi q(0)\sin\left(\sqrt{\xi\phi}\,t\right)}{\sqrt{\xi\phi}} \tag{265}$$

- Obtain $q(t)$ from $\dot{z}(t) = -\phi q(t)$:

$$\begin{aligned}q(t) &= \frac{1}{\phi}\dot{z}(t) \\ &= \frac{1}{\phi}\left[-\sqrt{\xi\phi}\,z(0)\sin\left(\sqrt{\xi\phi}\,t\right) - \phi q(0)\cos\left(\sqrt{\xi\phi}\,t\right)\right]\end{aligned} \tag{266}$$

10.3 Solving the model as a second-order differential equation

Note that the model can also be expressed in terms of a second-order differential equation in $z(t)$:

$$\ddot{z}(t) + \xi\phi z(t) = 0. \tag{267}$$

Solving this equation should obviously lead to the same final solution:

- Laplace transforms:

$$\big((s^2\bar{z} - sz(0) - \dot{z}(0)\big) + \xi\phi\bar{z} = 0. \tag{268}$$

- Solve for $\bar{z}$:

$$\bar{z} = \frac{sz(0) + \dot{z}(0)}{s^2 + \xi\phi}. \tag{269}$$

- Inverse transforms:

$$z(t) = z(0)\cos\left(\sqrt{\xi\phi}\,t\right) + \frac{\dot{z}(0)\sin\left(\sqrt{\xi\phi}\,t\right)}{\sqrt{\xi\phi}} \tag{270}$$

- Obtain $q(t)$ from $\dot{z}(t) = -\phi q(t)$:

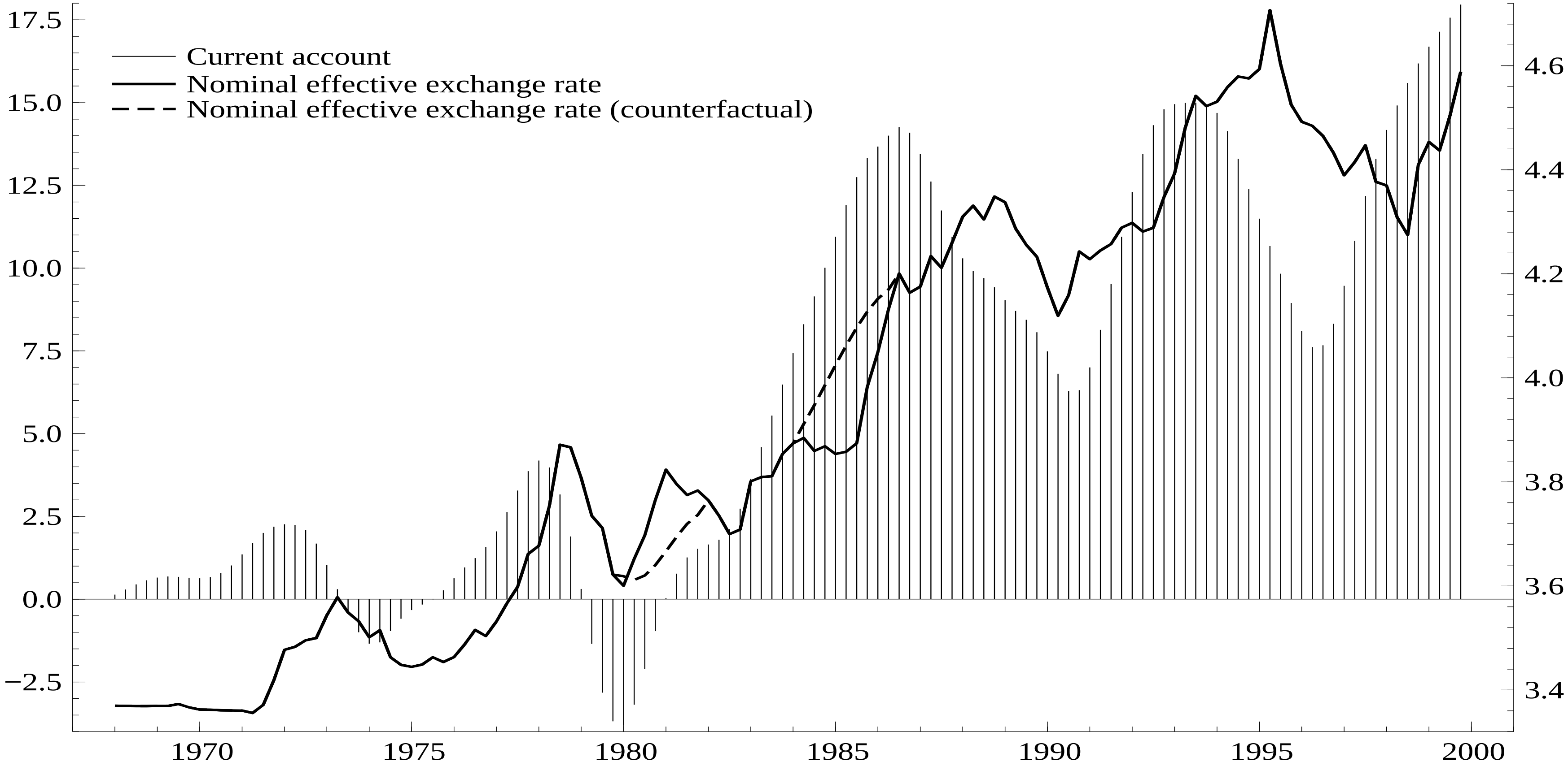
$$\begin{aligned} q(t) &= \frac{1}{\phi}\dot{z}(t) \\ &= \frac{1}{\phi}\left[-\sqrt{\xi\phi}\, z(0)\sin\left(\sqrt{\xi\phi}\, t\right) + \dot{z}(0)\cos\left(\sqrt{\xi\phi}\, t\right)\right] \end{aligned} \tag{271}$$

In this model,

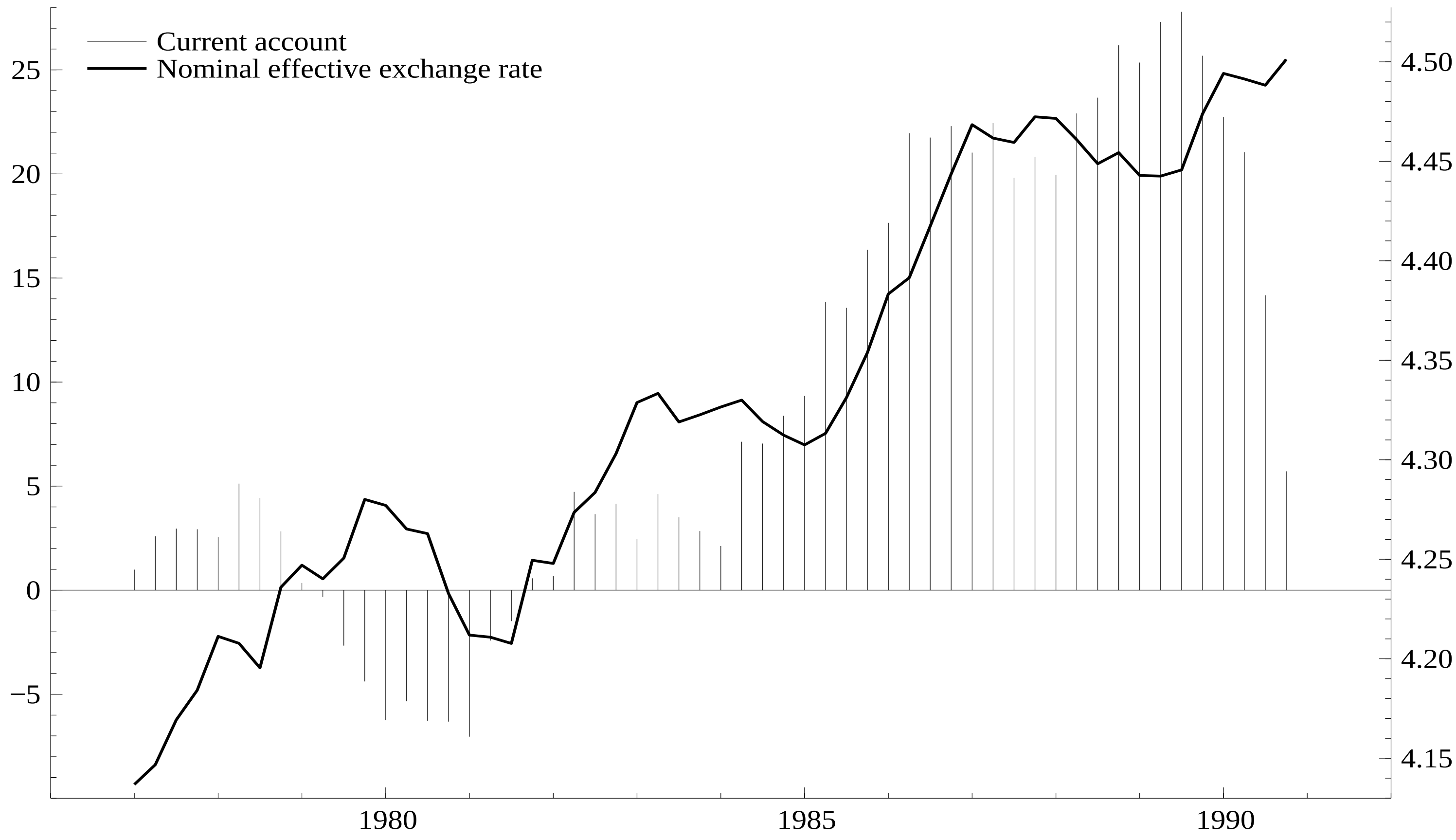
- the cyclical fluctuations come about since current account imbalances immediately produce offsetting payment flows,
- which push the exchange rate either up or down (depending on whether the current account is in surplus or in deficit).

Any current account imbalance thus carries with it the seed of its own reversal.

Now compare the model's prediction with the swings in Japan's and Germany's current account and exchange rate data plotted in the following two figures.



Japanese current account and counterfactual exchange rate.



German current account and nominal exchange rate in the 1980s.

Intertemporal optimization

11 Methods of intertemporal optimization

- Ordinary maximization
- Calculus of variations
- Optimal control
- Dynamic programming

12 Intertemporal approach to the current account

See Obstfeld and Rogoff (1995, 1996).

12.1 Current account

Current account balance of a country:

- exports minus imports of goods and services (elasticities approach), rents on labour and capital, unilateral transfers;
- increase in residents' claims on foreign incomes or outputs less increase in similar foreign-owned claims on home income or output;
- national saving less domestic investment (absorption approach, intertemporal approach).

The intertemporal approach views the current account balance as the outcome of forward-looking dynamic saving and investment decisions.

12.2 A one-good model with representative national residents

Consider a small open economy that produces and consumes a single composite good and trades freely with the rest of the world.

The current account, CA_t, is equal to the accumulation of net foreign assets and to the saving-investment balance:

$$CA_t = B_{t+1} - B_t = r_t B_t + Y_t - C_t - G_t - I_t, \tag{272}$$

where

$$\begin{aligned} B_{t+1} &:= \text{stock of net foreign assets at the end of period } t, \\ r_t &:= \text{interest rate}, \\ Y_t &:= \text{domestic output}, \\ C_t &:= \text{private consumption}, \\ G_t &:= \text{government consumption}, \\ I_t &:= \text{net investment}. \end{aligned} \tag{273}$$

13 Ordinary maximization by taking derivatives

13.1 Two-period model of international borrowing and lending

Utility:

$$U_1 = u(C_1) + \beta u(C_2). \tag{274}$$

Intertemporal budget constraint:

$$Y_1 + (1 + r)B_1 = C_1 + B_2, \tag{275}$$
$$Y_2 + (1 + r)B_2 = C_2 + B_3. \tag{276}$$

Current account:

$$CA_1 = S_1 = B_2 - B_1 = Y_1 + rB_1 - C_1, \tag{277}$$
$$CA_2 = S_2 = B_3 - B_2 = Y_2 + rB_2 - C_2. \tag{278}$$

Combining the intertemporal budget constraints yields:

$$Y_1 + \frac{1}{1+r}Y_2 + (1+r)B_1 = C_1 + \frac{1}{1+r}C_2 + \frac{1}{1+r}B_3 \tag{279}$$

$$\Leftrightarrow \quad C_2 = (1+r)Y_1 + Y_2 + (1+r)^2 B_1 - (1+r)C_1 - B_3 \tag{280}$$

$$\Leftrightarrow \quad C_1 = Y_1 + \frac{1}{1+r}Y_2 + (1+r)B_1 - \frac{1}{1+r}C_2 - \frac{1}{1+r}B_3. \tag{281}$$

Maximization problem:

$$\max_{C_1} = u(C_1) + \beta u((1+r)Y_1 + Y_2 + (1+r)^2 B_1 - (1+r)C_1 - B_3) \tag{282}$$

First-order condition:

$$u'(C_1) - \beta(1+r)u'(C_2) = 0 \quad \Leftrightarrow \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}. \tag{283}$$

Maximization problem:

$$\max_{C_2} = u\left(Y_1 + \frac{1}{1+r}Y_2 + (1+r)B_1 - \frac{1}{1+r}C_2 - \frac{1}{1+r}B_3\right) + \beta u(C_2) \tag{284}$$

First-order condition:

$$-\frac{1}{1+r}u'(C_1) - \beta u'(C_2) = 0 \quad \Leftrightarrow \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}. \tag{285}$$

Maximization problem:

$$\max_{B_2} = u(Y_1 + (1+r)B_1 - B_2) + \beta u(Y_2 + (1+r)B_2 - B_3) \tag{286}$$

First-order condition:

$$-u'(C_1) + \beta(1+r)u'(C_2) = 0 \quad \Leftrightarrow \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}. \tag{287}$$

Let $B_1 = B_3 = 0$. Let $u(\cdot) = \ln(\cdot)$. Then:

$$C_1 = \frac{1}{\beta(1+r)}C_2 = \frac{1}{1+\beta}\left(Y_1 + \frac{1}{1+r}Y_2\right), \tag{288}$$

$$C_2 = \frac{\beta}{1+\beta}\left((1+r)Y_1 + Y_2\right). \tag{289}$$

Current accounts:

$$CA_1 = B_2 - B_1 = \frac{\beta}{1+\beta}Y_1 - \frac{1}{(1+\beta)(1+r)}Y_2, \tag{290}$$

$$CA_2 = B_3 - B_2 = -\frac{\beta}{1+\beta}Y_1 + \frac{1}{(1+\beta)(1+r)}Y_2. \tag{291}$$

13.2 Digression on utility functions

13.2.1 Logarithmic utility.

Logarithmic utility is given by:

$$
\begin{aligned}
u(C) &= \ln(C), \\
u'(C) &= \frac{1}{C} > 0, \\
u''(C) &= -\frac{1}{C^2} < 0, \\
\bar{\gamma}(C) &= -\frac{u''(C)}{u'(C)} = \frac{1}{C}, \\
\gamma(C) &= -\frac{\frac{du'(C)}{u'(C)}}{\frac{dC}{C}} = -\frac{Cu''(C)}{u'(C)} = 1, \\
\sigma(C) &= \frac{1}{\gamma(C)} = -\frac{\frac{dC}{C}}{\frac{du'(C)}{u'(C)}} = -\frac{u'(C)}{Cu''(C)} = 1,
\end{aligned} \tag{292}
$$

where

- $\bar{\gamma}(C)$ is the absolute risk aversion, or the reciprocal of the so-called risk tolerance,
- $\gamma(C)$ is the relative risk aversion, or consumption elasticity of marginal utility, and
- $\sigma(C)$ is the elasticity of intertemporal substitution.

13.2.2 Isoelastic utility.

The class of period utility functions characterized by a constant elasticity of intertemporal substitution is:

$$\begin{aligned}
u(C) &= \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \qquad \sigma > 0, \\
u'(C) &= C^{-\frac{1}{\sigma}} > 0, \\
u''(C) &= -\frac{1}{\sigma}C^{-\frac{1}{\sigma}-1} < 0, \\
\bar{\gamma}(C) &= -\frac{u''(C)}{u'(C)} = \frac{1}{\sigma C}, \\
\gamma(C) &= -\frac{Cu''(C)}{u'(C)} = \frac{1}{\sigma} = \gamma = \text{const.} \\
\sigma(C) &= \frac{1}{\gamma(C)} = \sigma = \text{const.}
\end{aligned} \tag{293}$$

For $\sigma = 1$, the isoelastic utility function is replaced by its limit, $\ln(C)$.

13.2.3 Linear-quadratic utility.

Linear-quadratic utility is given by:

$$\begin{aligned}
u(C) &= -\frac{(b-C)^2}{2}, \\
u'(C) &= b - C, \qquad u'(C) \geq 0 \text{ if } C \leq b, \\
u''(C) &= -1 < 0, \\
\bar{\gamma}(C) &= -\frac{u''(C)}{u'(C)} = \frac{1}{b-C}, \\
\gamma(C) &= -\frac{Cu''(C)}{u'(C)} = \frac{C}{b-C}, \\
\sigma(C) &= \frac{1}{\gamma(C)} = \frac{b}{C} - 1.
\end{aligned} \tag{294}$$

13.2.4 Exponential utility.

Exponential utility is given by:

$$\begin{aligned}
u(C) &= -be^{-\frac{C}{b}}, \qquad b > 0, \\
u'(C) &= e^{-\frac{C}{b}} > 0, \\
u''(C) &= -\frac{1}{b}e^{-\frac{C}{b}} < 0, \\
\bar{\gamma}(C) &= -\frac{u''(C)}{u'(C)} = \frac{1}{b} = \text{const.}, \\
\gamma(C) &= -\frac{Cu''(C)}{u'(C)} = \frac{C}{b}, \\
\sigma(C) &= \frac{1}{\gamma(C)} = \frac{b}{C}.
\end{aligned} \tag{295}$$

13.2.5 The HARA class of utility functions

A HARA utility function, $u(C)$, is one whose absolute risk aversion is hyperbolic:

$$\bar{\gamma} = -\frac{u''(C)}{u'(C)} = \frac{1}{aC + b} > 0, \tag{296}$$

for some constants a and b. Since the inverse of absolute risk aversion is risk tolerance, a HARA utility function exhibits linear risk tolerance:

$$\frac{1}{\bar{\gamma}} = -\frac{u'(C)}{u''(C)} = aC + b > 0. \tag{297}$$

The relative risk aversion of a HARA utility function is given by:

$$\gamma(C) = -\frac{Cu''(C)}{u'(C)} = \frac{C}{aC + b} = \frac{1}{a} - \frac{b}{a^2C + ab}. \tag{298}$$

For a utility function of the HARA class:

- risk tolerance (the reciprocal of absolute risk aversion) is a linearly increasing function of a and is constant if $a = 0$;
- relative risk aversion is rising with b and is constant if $b = 0$.

It can be shown that all the utility functions mentioned above belong to the HARA family:

Utility function	a	b
Logarithmic	> 0	$= 0$
Isoelastic	> 0	$= 0$
Linear-quadratic	< 0	> 0
Exponential	$= 0$	> 0

13.3 Two-period model with investment

Production function:

$$Y = F(K). \tag{299}$$

As usual, $F'(K) > 0$, $F''(K) < 0$ and $F(0) = 0$.

Utility:

$$U_1 = u(C_1) + \beta u(C_2). \tag{300}$$

Intertemporal budget constraint, where $I_t = K_{t+1} - K_t$:

$$Y_1 + (1 + r)B_1 + K_1 = C_1 + B_2 + K_2, \tag{301}$$

$$Y_2 + (1 + r)B_2 + K_2 = C_2 + B_3 + K_3. \tag{302}$$

Current account:

$$CA_1 = S_1 - I_1 = B_2 - B_1 = Y_1 + rB_1 - C_1 - I_1, \tag{303}$$

$$CA_2 = S_2 - I_2 = B_3 - B_2 = Y_2 + rB_2 - C_2 - I_2. \tag{304}$$

Combining the intertemporal budget constraints yields:

$$Y_1 + \frac{1}{1+r}Y_2 + (1+r)B_1 + K_1 = C_1 + \frac{1}{1+r}C_2 + \frac{1}{1+r}B_3 + \frac{1}{1+r}K_3 \tag{305}$$

$$\Leftrightarrow \quad C_2 = (1+r)Y_1 + Y_2 + (1+r)^2 B_1 + (1+r)K_1 - (1+r)C_1 - B_3 - K_3 \tag{306}$$

$$\Leftrightarrow \quad C_1 = Y_1 + \frac{1}{1+r}Y_2 + (1+r)B_1 - \frac{1}{1+r}C_2 - \frac{1}{1+r}B_3 - \frac{1}{1+r}K_3. \tag{307}$$

Maximization problem:

$$\max_{B_2,K_2} = u(F(K_1) + (1+r)B_1 + K_1 - B_2 - K_2) + \beta u(F(K_2) + (1+r)B_2 + K_2 - B_3 - K_3) \tag{308}$$

First-order conditions:

$$-u'(C_1) + \beta(1+r)u'(C_2) = 0 \quad \Leftrightarrow \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}, \tag{309}$$

$$-u'(C_1) + \beta(1+F'(K_2))u'(C_2) = 0. \quad \Leftrightarrow \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+F'(K_2)}. \tag{310}$$

$$\tag{311}$$

Therefore returns on capital and foreign assets must be equal:

$$F'(K_2) = r. \tag{312}$$

Let $B_1 = B_3 = 0$. Let $u(\cdot) = \ln(\cdot)$. Then:

$$C_1 = \frac{1}{\beta(1+r)} C_2 = \frac{1}{1+\beta} \left(Y_1 + \frac{1}{1+r} Y_2 \right), \tag{313}$$

$$C_2 = \frac{\beta}{1+\beta} \left((1+r) Y_1 + Y_2 \right). \tag{314}$$

13.4 An infinite-horizon model

Utility at time t:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s). \tag{315}$$

Intertemporal budget constraint:

$$A_s F(K_s) + (1+r)B_s + K_s = C_s + B_{s+1} + K_{s+1} + G_s \tag{316}$$

The infinite-horizon budget constraint is:

$$\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s + (1+r)B_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (C_s + I_s + G_s). \tag{317}$$

Here it is assumed that the transversality condition holds:

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^{T} B_{t+T+1} = 0. \tag{318}$$

Maximization problem:

$$\max_{B_{s+1},K_{s+1}} \sum_{s=t}^{\infty} \beta_{s-t} u[A_s F(K_s) + (1+r)B_s + K_s - B_{s+1} - K_{s+1} - G_s] \tag{319}$$

First-order conditions:

$$-u'(C_s) + \beta(1+r)u'(C_{s+1}) = 0 \tag{320}$$

$$\Leftrightarrow \quad \frac{\beta u'(C_{s+1})}{u'(C_s)} = \frac{1}{1+r}, \tag{321}$$

$$-u'(C_s) + \beta(1 + A_{s+1}F'(K_{s+1}))u'(C_{s+1}) = 0 \tag{322}$$

$$\Leftrightarrow \quad \frac{\beta u'(C_{s+1})}{u'(C_s)} = \frac{1}{1 + A_{s+1}F'(K_{s+1})}. \tag{323}$$

Therefore returns on capital and foreign assets must be equal:

$$A_{s+1}F'(K_{s+1}) = r. \tag{324}$$

Note that when $\beta = 1/(1+r)$, optimal consumption is constant:

$$C_t = \frac{r}{1+r}\left[(1+r)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s - I_s)\right]. \tag{325}$$

If the period utility function is isoelastic, the Euler equation (320) takes the form

$$C_{s+1} = (1+r)^{\sigma}\beta^{\sigma}C_s. \tag{326}$$

We can use it to eliminate C_{t+1}, C_{t+2}, etc. from budget constraint (317). Under the assumption that $(1+r)^{\sigma-1}\beta^{\sigma} < 1$, so that consumption grows a a net rate below r, the result is the consumption function

$$C_t = \frac{(1+r)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - I_s - G_s)}{\sum_{s=t}^{\infty}[(1+r)^{\sigma-1}\beta^{\sigma}]^{s-t}}. \tag{327}$$

Defining $\theta \equiv 1 - (1+r)^{\sigma}\beta^{\sigma}$, we rewrite this as:

$$C_t = \frac{r+\theta}{1+r}\left[(1+r)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - I_s - G_s)\right]. \tag{328}$$

Given r, consumption is a decreasing function of β.

13.5 Dynamics of the current account

See Obstfeld and Rogoff (1996, section 2.2).

For a constant interest rate r, define the permanent level of a variable X on date t by:

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\tilde{X}_t=\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}X_s, \tag{329}$$

so that

$$\tilde{X}_t=\frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}X_s. \tag{330}$$

The permanent level of X, $\tilde{X}$, is its annuity value at the prevailing interest rate.

Using (325), we obtain:

$$\begin{aligned}CA_t&=B_{t+1}-B_t\\&=Y_t+rB_t-C_t-I_t-G_t\\&=(Y_t-\tilde{Y}_t)-(I_t-\tilde{I}_t)-(G_t-\tilde{G}_t).\end{aligned} \tag{331}$$

When $\beta \neq 1/(1+r)$ and utility is isoelastic,

$$CA_t = (Y_t - \tilde{Y}_t) - (I_t - \tilde{I}_t) - (G_t - \tilde{G}_t) - \frac{\theta}{1+r} W_t, \tag{332}$$

where

$$W_t = (1+r)B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} (Y_s - I_s - G_s) \tag{333}$$

and $\theta = 1 - (1+r)^{\sigma} \beta^{\sigma}$.

13.6 A model with consumer durables

See Obstfeld and Rogoff (1996, section 2.4).

Let C_s be consumption of nondurables and D_s be the stock of durable goods the consumer owns as date s ends. A stock of durables yields its owner a proportional service flow each period it is owned. Let p be the price of durable goods in terms of nondurable consumption (determined in the world market).

Utility function

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t}[\gamma \ln C_s + (1-\gamma) \ln D_s] \tag{334}$$

Period-to-period budget constraint:

$$F(K_s) + (1+r_s)B_s + K_s + p_s(1-\delta)D_{s-1} = C_s + B_{s+1} + K_{s+1} + p_s D_s + G_s, \tag{335}$$

where $p_s[D_s - (1-\delta)D_{s-1}]$ is the cost of durable goods purchases in period s.

Maximization problem:

$$\max_{B_{s+1}, K_{s+1}, D_s} \sum_{s=t}^{\infty} \beta^{s-t}[\gamma \ln C_s + (1-\gamma) \ln D_s], \tag{336}$$

where $C_s = F(K_s) - [B_{s+1} - (1 + r_s)B_s] - (K_{s+1} - K_s) - p_s[D_s - (1 - \delta)D_{s-1}] - G_s$.

First-order conditions:

$$-\frac{1}{C_s} + \beta(1 + r_s)\frac{1}{C_{s+1}} = 0, \tag{337}$$

$$-\frac{1}{C_s} + \beta(1 + F'(K_{s+1})\frac{1}{C_{s+1}} = 0, \tag{338}$$

$$\frac{1 - \gamma}{D_s} - \frac{\gamma p_s}{C_s} + \frac{\beta\gamma p_{s+1}(1 - \delta)}{C_{s+1}} = 0. \tag{339}$$

Rewrite these equations:

$$r_s = F'(K_{s+1}), \tag{340}$$

$$C_{s+1} = \beta(1 + r_s)C_s, \tag{341}$$

$$\frac{(1 - \gamma)C_s}{\gamma D_s} = p_s - \frac{1 - \delta}{1 + r_{s+1}}p_{s+1} \equiv \iota_s. \tag{342}$$

Here, ι is the implicit date s rental price, or user cost, of the durable good, that is, the net expense of buying the durable in one period, using it in the same period, and selling it in the next. Equation (342) states that, at an optimum, the marginal rate of substitution of nondurables consumption for the services of durables equals the user cost of durables in terms of nondurables consumption.

Intertemporal budget constraint (with constant r):

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(C_s+\iota_s D_s)= \\ (1+r)B_t+(1-\delta)p_t D_{t-1}+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s-G_s-I_s). \quad (343)$$

This constraint states that the present value of expenditures (the sum of nondurables purchases plus the implicit rental cost of the durables held) equals initial financial assets (including durables) plus the present value of net output.

Assuming that $\beta=1/(1+r)$ (so that nondurables and durables consumption is constant):

$$C_t=\frac{\gamma r}{1+r}\left[(1+r)B_t+(1-\delta)p_t D_{t-1}+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s-G_s-I_s)\right] \quad (344)$$

$$D_t=\frac{(1-\gamma)r}{\iota(1+r)}\left[(1+r)B_t+(1-\delta)p_t D_{t-1}+\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s-G_s-I_s)\right] \quad (345)$$

How do durables affect the current account? With p constant:

$$p=\left(\frac{1+r}{1+\delta}\right)\left(\frac{1-\gamma}{\gamma}\right)\frac{C_s}{D_s}, \qquad s\geq t. \quad (346)$$

Let $Z = Y - G - I$. Then:

$$\begin{aligned} CA_t = B_{t+1} - B_t &= rB_t + Z_t - \frac{C_t}{\gamma} - \frac{(1-\gamma)C_t}{\gamma} - p[D_t - (1-\delta)D_{t-1}] \\ &= \left[Z_t - \frac{r}{1+r} \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} Z_s \right] \\ &\quad - \frac{1-\delta}{1+r} pD_{t-1} + \frac{(1-\gamma)C_t}{\gamma} - pD_t + (1-\delta)pD_{t-1} \\ &= (Y_t - \tilde{Y}_t) - (I_t - \tilde{I}_t) - (G_t - \tilde{G}_t) + (\iota - p)\Delta D_t. \end{aligned} \tag{347}$$

13.7 Firms, the labour market and investment

See Obstfeld and Rogoff (1996, section 2.5.1).

The production function (homogeneous to degree one) is $AF(K, L)$, where L is constant. We can think of output as being produced by a single representative domestic firm that behaves competitively and is owned entirely by domestic residents. V_t is the date t price of a claim to the firm's entire future profits (starting on date $t + 1$). Let x_{s+1} be the share of the domestic firm owned by the representative consumer at the end of date s and d_s the dividends the firm issues on date s.

13.7.1 The consumer's problem.

Utility function:

$$U_t = \sum_{s=t}^{\infty} u(C_s). \tag{348}$$

Period-to-period financial constraint:

$$B_{s+1} - B_s + V_s x_{s+1} - V_{s-1} x_s = rB_s + d_s x_s + (V_s - V_{s-1})x_s + w_s L - C_s - G_s. \tag{349}$$

Maximization problem:

$$\max_{B_{s+1},x_{s+1}} \sum_{s=t}^{\infty} u[(1+r)B_s - B_{s+1} - V_s(x_{s+1} - x_s) + d_s x_s + w_s L - G_s]. \tag{350}$$

First-order conditions:

$$u'(C_s) = (1+r)\beta u'(C_{s+1}), \tag{351}$$

$$V_s u'(Cs) = (V_{s+1} + d_{s+1})\beta u'(C_{s+1}). \tag{352}$$

From this, we see that returns on foreign bonds and shares must be equal:

$$1 + r = \frac{d_{s+1} + V_{s+1}}{V_s} \tag{353}$$

A useful reformulation of the individual's budget constraint uses the variable Q_{s+1}, which is the value of the individual's financial wealth at the end of period s:

$$Q_{s+1} = B_{s+1} + V_s x_{s+1}. \tag{354}$$

The period-to-period financial constraint becomes:

$$Q_{s+1} - Q_s = rQ_s + w_s L - C_s - G_s, \qquad s = t+1, t+2, \ldots, \tag{355}$$

$$Q_{t+1} = (1+r)B_t + d_t x_t + V_t x_t + w_t L - C_t - G_t. \tag{356}$$

By forward iteration, we obtain:

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}C_s=(1+r)B_t+d_tx_t+V_tx_t+\sum_{s=t}^{\infty}\left(\frac{r}{1+r}\right)^{s-t}(w_sL-G_s). \tag{357}$$

Here it is supposed that the following transversality condition holds:

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T}Q_{t+T+1}=0. \tag{358}$$

13.7.2 The stock market value of the firm.

Note that equation (353) implies:

$$\begin{aligned}V_t&=\frac{d_{t+1}}{1+r}+\frac{V_t+1}{1+r}\\&=\sum_{t+1}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}d_s.\end{aligned} \tag{359}$$

We rule out self-fulfilling speculative asset-price bubbles:

$$\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^{T}V_{t+T}=0. \tag{360}$$

13.7.3 Firm behaviour.

The dividends a firm pays out in a period are its current profits less investment expenditure, that is, $d_s = Y_s - w_s L_s - I_s$. The value of the firm can therefore be written as follows:

$$V_t = \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} [A_s F(K_s, L_s) - w_s L_s - (K_{s+1} - K_s)]. \tag{361}$$

The firm maximizes the present value of current and future dividends, given K_t:

$$d_t + V_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} [A_s F(K_s, L_s) - w_s L_s - (K_{s+1} - K_s)]. \tag{362}$$

First-order conditions for capital and labour:

$$A_s F_K(K_s, L_s) = r, \qquad s > t, \tag{363}$$

$$A_s F_L(K_s, L_s) = w_s, \qquad s \geq t. \tag{364}$$

Note that the consumer's problem is the same is the same as in an economy without a firm, where the consumer is itself the producer. This is so since the Euler equation and budget constraints are identical, provided the equilibrium conditions $x_s = 1$ and $L_s = L$ hold on all dates s. To see this, combine equations (361) and (351).

13.8 Investment when capital is costly to install: Tobin's q

See Obstfeld and Rogoff (1996, section 2.5.2).

There are now quadratic installation costs for investment. The firm maximizes, for a given K_t:

$$d_t + V_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left[A_s F(K_s, L_s) - \frac{\chi}{2} \frac{I_s^2}{K_s} - w_s L_s - I_s \right], \tag{365}$$

subject to

$$K_{s+1} - K_s = I_s. \tag{366}$$

Lagrangian, to differentiate with respect to labour, investment and capital:

$$\mathcal{L}_t = \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \left[A_s F(K_s, L_s) - \frac{\chi}{2} \frac{I_s^2}{K_s} - w_s L_s - I_s - q_s (K_{s+1} - K_s - I_s) \right]. \tag{367}$$

First-order conditions:

$$A_s F_L(K_s, L_s) - w = 0, \tag{368}$$

$$-\frac{\chi I_s}{K_s} - 1 + q_s = 0, \tag{369}$$

$$-q_s + \frac{A_{s+1} F_K(K_{s+1}, L_{s+1}) + \frac{\chi}{2}(I_{s+1}/K_{s+1})^2 + q_{s+1}}{1 + r} = 0. \tag{370}$$

Dynamic optimization in continuous time

14 Optimal control theory

14.1 Deriving the fundamental results using an economic example

A firm wishes to maximize its total profits over some period of time starting at date $t = 0$ and ending at date T:

$$W(k_0, \boldsymbol{x}) = \int_0^T u(k, x, t)dt$$

The variable k_t is the capital stock at date t, the initial capital stock k_0 is given to the firm. The variable x_t is chosen by the firm at every instant. The function $u(k_t, x_t, t)$ determines the rate at which profits are being earned at time t as a result of having capital k and taking decisions x. The integral $W(\cdot)$ sums up the profits that are being earned at all the instants from the initial date until date T, when starting with a capital stock k_0 and following the decision policy $\boldsymbol{x}$. Note that $\boldsymbol{x}$ denotes a time path for the decision variable x, it comprises all the decisions taken at all the instants between date zero and T.

Changes in the capital stock are governed by the following equation:

$$\dot{k} = \frac{dk}{dt} = f(k, x, t)$$

Note that the decisions x influence not only contemporaneous profits but also the rate at which the capital stock is changing and thereby the amount of capital available in the future. As we will see, this gives rise to a potential tradeoff.

The problem is how to choose the time path $\boldsymbol{x}$ so as to maximize the overall result, W. Difficult to solve, since it involves optimization in a dynamic context. Ordinary calculus only tells us how to choose individual variables to solve an optimization problem. How to solve the problem then? Reduce the problem to one to which ordinary calculus can be applied.

Consider the problem when starting at date t:

$$\begin{aligned} W(k_t, \boldsymbol{x}, t) &= \int_t^T u(k_\tau, x_\tau, \tau) d\tau \\ &= u(k_t, x_t, t)\Delta + \int_{t+\Delta}^T u(k_\tau, x_\tau, \tau) d\tau \\ &= u(k_t, x_t, t)\Delta + W(k_{t+\Delta}, \boldsymbol{x}, t+\Delta) \end{aligned}$$

where Δ is a very short time interval. This says that the value contributed to the total sum of profits from date t on is made up of two parts. The first part consists of the profits accrued in the short time interval beginning at date t. The second part is the sum of all the profits earned from date $t+\Delta$.

Let $V^*(k_t, t)$ denote the best achievable value for W when starting at date t with a capital stock k_t:

$$V^*(k_t, t) \equiv \max_{\boldsymbol{x}} W(k_t, \boldsymbol{x}, t)$$

Suppose the firm chooses x_t (any decision, not necessarily the optimal one) for the short, initial time interval Δ and thereafter follows the best possible policy. Then this would yield:

$$V(k_t, x_t, t) = u(k_t, x_t, t)\Delta + V^*(k_{t+\Delta}, t+\Delta) \tag{371}$$

Now the whole problem reduces to finding the optimal value for x_t. Adopting this value would make V in the last equation become equal to V^*. The first-order condition is:

$$\begin{aligned} & \Delta\frac{\partial}{\partial x_t}u(k, x_t, t) + \frac{\partial}{\partial x_t}V^*(k_{t+\Delta}, t+\Delta) \\ = \;& \Delta\frac{\partial}{\partial x_t}u(k, x_t, t) + \frac{\partial V^*}{\partial k_{t+\Delta}}\frac{\partial k_{t+\Delta}}{\partial x_t} \\ = \;& 0 \end{aligned} \tag{372}$$

Consider the second factor of the second term. Since Δ is small,

$$k_{t+\Delta} = k_t + \dot{k}\Delta$$

Remember that $\dot{k}$, the rate at which the capital stock changes, depends on the decision variable, x_t:

$$\dot{k} = f(k, x, t)$$

We obtain:

$$\frac{\partial k_{t+\Delta}}{\partial x_t} = \Delta \frac{\partial f}{\partial x_t}$$

What is the meaning of the first factor, $\partial V^*/\partial k$? It is the marginal value of capital at date $t + \Delta$, telling us how the maximal value of W changes in response to a unit increase in the capital stock at date $t + \Delta$. Let it be denoted by λ_t:

$$\lambda_t \equiv \frac{\partial}{\partial k} V^*(k, t)$$

λ_t will sometimes be referred to as the co-state variable whereas k_t is the state variable.

The first-order condition in equation (372) now becomes:

$$\begin{aligned}&\frac{\partial u}{\partial x_t} + \lambda_{t+\Delta}\frac{\partial f}{\partial x_t} \\ &= \frac{\partial u}{\partial x_t} + \lambda_t\frac{\partial f}{\partial x_t} + \dot{\lambda}_t\Delta\frac{\partial f}{\partial x_t} \\ &= 0\end{aligned}$$

Here use has been made of the fact that the marginal value of capital changes smoothly over time so that $\lambda_{t+\Delta} = \lambda_t + \dot{\lambda}\Delta$.

Now let Δ approach zero. The third term becomes negligible, and we obtain the following important result:

$$\frac{\partial u}{\partial x_t} + \lambda\frac{\partial f}{\partial x_t} = 0 \tag{373}$$

It says that along the optimal path, the marginal short-run effect of a change in the decision variable must just offset the long-run effect of that change on the total value of the capital stock.

Suppose the optimal value for x_t has been determined by equation (373) and let it be denoted by x_t^*. When this decision is chosen, $V(k_t, x_t, t)$ in equation (371) becomes equal to $V^*(k_t, t)$:

$$V^*(k,t) = u(k, x_t^*, t)\Delta + V^*(k_{t+\Delta}, t+\Delta)$$

Let us differentiate this with respect to k:

$$\begin{aligned}\lambda_t &= \Delta\frac{\partial u}{\partial k} + \frac{\partial}{\partial k}V^*(k_{t+\Delta}, t+\Delta) \\ &= \Delta\frac{\partial u}{\partial k} + \frac{\partial k_{t+\Delta}}{\partial k}\lambda_{t+\Delta} \\ &= \Delta\frac{\partial u}{\partial k} + \left(1 + \Delta\frac{\partial f}{\partial k}\right)(\lambda + \dot{\lambda}\Delta) \\ &= \Delta\frac{\partial u}{\partial k} + \lambda + \Delta\lambda\frac{\partial f}{\partial k} + \Delta\dot{\lambda} + \dot{\lambda}\frac{\partial f}{\partial k}\Delta^2\end{aligned}$$

The final term in the last line becomes negligible when Δ approaches zero; thus we ignore it. After rearranging, we have another important result:

$$-\dot{\lambda} = \frac{\partial u}{\partial k} + \lambda\frac{\partial f}{\partial k} \tag{374}$$

This means that when the optimal path of capital accumulation is followed, the rate at which a unit of capital depreciates in a short time interval must be equal to both its contribution to profits during the interval and its contribution to potential profits in the future, that is, after the end of the interval.

14.2 The Maximum Principle

The two results in equations (373) and (374), as well as the requirement that $\dot{k} = f(k, x, t)$ which is part of the problem setting, are conveniently summarized with the aid of an auxiliary function, the so-called Hamiltonian function:

$$H \equiv u(k, x, t) + \lambda_t f(k, x, t)$$

All three formulas can be expressed in terms of partial derivatives of the Hamiltonian:

$$\frac{\partial H}{\partial \lambda} = \dot{k} \quad (375)$$
$$\frac{\partial H}{\partial x} = 0 \quad (376)$$
$$\frac{\partial H}{\partial k} = -\dot{\lambda} \quad (377)$$

These three formulas jointly determine the time path of the decision, or choice, variable x_t, the capital stock k_t and the value of capital λ_t.

Why the name Maximum Principle? This can be seen by interpreting the Hamiltonian itself, but the same conclusions can be derived by looking at a slightly modified Hamiltonian:

$$\begin{aligned} H^* &\equiv u(k,x,t) + \frac{d}{dt}\lambda k \\ &= u(k,x,t) + \lambda \dot{k} + \dot{\lambda} k \end{aligned}$$

H^* can be interpreted as the sum of profits realized during a given instant of time and the change in the value of the capital stock (resulting both from quantity and valuation changes) during that instant. In other words, it summarizes all current and potential future profits. This is what we want to maximize throughout the period considered, from the initial date to date T. But if we maximize H^* with respect to x and k, we just obtain equations (373) and (374). The modified Hamiltonian H^* differs from H in that it includes capital gains. But this is just a matter of definition. When we use H instead of H^*, the relevant formulas are those in equations (375) to (377).

For boundary conditions, see Chiang (1992). Second-order conditions haven't been mentioned but should in principle also be checked.

14.3 Standard formulas

In economics, the optimal control problem often takes the following form. The objective function is given by:

$$W(k_0) = \int_0^t u(k_t, x_t, t)e^{-\rho t}dt$$

where $e^{-\rho t}$ is the discount factor and k_0, the initial value of the state variable, is a given as part of the problem. The state equation is given by $\dot{k} = f(k_t, u_t, t)$.

The Hamiltonian for such a problem is:

$$H \equiv u(k_t, x_t, t)e^{-\rho t} + \lambda_t f(k_t, x_t, t)$$

The Maximum Principle demands that the following equations hold for $t \,\epsilon\, [0, \infty]$:

$$\begin{aligned}
\frac{\partial H}{\partial \lambda} &= \dot{k} \\
\frac{\partial H}{\partial x} &= 0 \\
\frac{\partial H}{\partial k} &= -\dot{\lambda}
\end{aligned}$$

An equally valid method is to use the current-value Hamiltonian, defined as:

$$H_C \equiv He^{-\rho t} = u(k_t, x_t, t) + \mu_t f(k_t, x_t, t)$$

where we work with a redefined co-state variable $\mu \equiv \lambda_t e^{\rho t}$. The first-order conditions now become:

$$\begin{aligned} \frac{\partial H}{\partial \mu} &= \dot{k} \\ \frac{\partial H}{\partial x} &= 0 \\ \frac{\partial H}{\partial k} &= -\dot{\mu} + \rho\mu \end{aligned}$$

14.4 Literature

A good introduction to optimal control theory is Chiang (1992). Sydsæter, Strøm and Berck (2000) provide a collection of useful formulas. The derivations above summarize Dorfman's (1969) article which gives an economic interpretation to the Maximum Principle.

15 Continuous-time stochastic processes

15.1 Wiener process

A Wiener process is a real-valued continuous-time stochastic process $\omega(t)$ with the following properties:

- The time path followed by $\omega(t)$ is continuous.
- The increments of $\omega(t)$ over infinitesimal time intervals dt are normally distributed with mean zero and variance dt:

  $$d\omega \sim \mathrm{N}(0, dt). \tag{378}$$

  This may also be written as:

  $$d\omega \sim \sqrt{dt}\,\varepsilon_t, \tag{379}$$

  where $\varepsilon_t \sim \mathrm{N}(0, 1)$.
- The increments of $\omega(t)$ over non-overlapping time intervals are independent.

Note that a Wiener process, although continuous, is not differentiable in general.

15.2 Ito process

An Ito process $s(t)$ is a real-valued continuous-time stochastic process with stochastic differential

$$ds = g(s,t)dt + \sigma(s,t)d\omega, \tag{380}$$

where ω is a Wiener process.

We shall call g the drift of the Ito process and σ its volatility.

Note:

- The time path followed by $s(t)$ is continuous.
- The increments of $s(t)$ over infinitesimal time intervals dt are normally distributed with mean $g(s,t)dt$ and variance $\sigma^2(s,t)dt$:

$$ds \sim \mathrm{N}(g(s,t)dt, \sigma^2(s,t)dt). \tag{381}$$

- The increments of $s(t)$ over non-overlapping time intervals are independent.

15.2.1 Absolute Brownian motion

Absolute Brownian motion is an Ito process $s(t)$ defined by the following stochastic differential equation:

$$ds = \mu dt + \zeta d\omega. \tag{382}$$

15.2.2 Geometric Brownian motion

Geometric Brownian motion (often used to model asset prices) is an Ito process $s(t)$ defined by the following stochastic differential equation:

$$\frac{ds}{s} = \mu dt + \zeta d\omega \qquad \Leftrightarrow \qquad ds = \mu s dt + \zeta s d\omega. \tag{383}$$

15.2.3 Ito calculus

When working with Ito processes, the basic product rules of Ito calculus prove useful:

$$dt^2 \approx 0 \quad (\text{since } dt^2 \ll dt \text{ as } dt \to 0), \tag{384}$$
$$dt d\omega \approx 0 \quad (\text{since } dt d\omega \ll dt \text{ as } dt \to 0), \tag{385}$$
$$d\omega^2 = \mathrm{Var}(d\omega) = dt. \tag{386}$$

15.3 Ito's lemma

Consider a stochastic process $y(s(t), t)$, where $s(t)$ is an Ito process and y is twice continuously differentiable. Ito's lemma asserts that $y(s(t), t)$ is also an Ito process with stochastic differential

$$dy = \left(y_t + g y_s + \frac{1}{2}\sigma^2 y_{ss} \right) dt + \sigma y_s d\omega, \tag{387}$$

where y_t, y_s and y_{ss} are the partial derivatives of y evaluated at (s, t) and g and σ are the drift and volatility of $s(t)$ evaluated at (s, t).

To see the intuition behind Ito's lemma, consider the Taylor series expansion of dy around (s, t):

$$\begin{aligned} dy &= y_t dt + y_s ds + \frac{1}{2}\left(y_{tt} dt^2 + 2 y_{st} ds dt + y_{ss} ds^2 \right) + \ldots \\ &= y_t dt + y_s (g dt + \sigma d\omega) + \frac{1}{2} y_{ss} \sigma^2 dt. \end{aligned} \tag{388}$$

where use is made of the fact that dt^2, $dsdt$ and all terms of order three and above are zero and that $ds^2 = \sigma^2 dt$.

15.4 Examples of Ito's lemma

15.4.1 Exponential function

Let $s(t)$ be an absolute Brownian motion:

$$ds = \mu dt + \zeta d\omega. \tag{389}$$

Now consider the exponential function of $s(t)$:

$$y(s) = e^s, \quad y'(s) = e^s, \quad y''(s) = e^s. \tag{390}$$

Hence:

$$\begin{aligned} dy &= \left(y'(s)\mu + \frac{1}{2} y''(s)\zeta^2 \right) dt + y'(s)\zeta d\omega \\ &= \left(e^s\mu + \frac{1}{2} e^s\zeta^2 \right) dt + e^s\zeta d\omega \\ &= \left(\mu + \frac{1}{2}\zeta^2 \right) y dt + \zeta y d\omega. \end{aligned} \tag{391}$$

Ito's lemma can thus be used to show that the exponential function of an absolute Brownian motion with drift μ and volatility ζ is a geometric Brownian motion with drift $(\mu + \zeta^2/2)y$ and volatility ζy.

15.4.2 Logarithm of a geometric Brownian motion process

Let $s(t)$ be a geometric Brownian motion process:

$$ds = \mu s dt + \zeta s d\omega. \tag{392}$$

Now consider the logarithm of $s(t)$:

$$y(s) = \ln(s), \quad y'(s) = \frac{1}{s}, \quad y'(s) = -\frac{1}{s^2}. \tag{393}$$

Hence:

$$\begin{aligned} dy &= \left(y'(s)\mu s + \frac{1}{2} y''(s)\zeta^2 s^2 \right) dt + y'(s)\zeta s d\omega \\ &= \left(\frac{1}{s}\mu s - \frac{1}{2}\frac{1}{s^2}\zeta^2 s^2 \right) dt + \frac{1}{s}\zeta s d\omega \\ &= \left(\mu - \frac{\zeta^2}{2} \right) dt + \zeta d\omega. \end{aligned} \tag{394}$$

Ito's lemma can thus be used to show that the logarithm of a geometric Brownian motion with drift μs and volatility ζs is an absolute Brownian motion with drift $\mu - \zeta^2/2$ and volatility ζ.

Integrating both sides of equation (394), we have:

$$\begin{aligned}\ln(s_t) - \ln(s_0) &= \int_0^t \left(\mu - \frac{\zeta^2}{2}\right) d\tau + \int_0^t \zeta d\omega \\ &= \left(\mu - \frac{\zeta^2}{2}\right) t + \zeta \omega_t.\end{aligned} \tag{395}$$

By taking exponentials on both sides of equation (395) and rearranging, we obtain a closed-form solution to the stochastic differential to the geometric Brownian motion process in equation (392):

$$s_t = s_0 e^{\left(\mu - \frac{\zeta^2}{2}\right) t + \zeta \omega_t} = s_0 e^{\left(\mu - \frac{\zeta^2}{2}\right) t} e^{\zeta \omega_t}. \tag{396}$$

15.4.3 Power function

Let $s(t)$ be a geometric Brownian motion:

$$ds = \mu s dt + \zeta s d\omega. \tag{397}$$

Now consider the power function of $s(t)$:

$$y(s) = As^{\beta}, \quad y'(s) = A\beta s^{\beta-1}, \quad y''(s) = A\beta(\beta-1)s^{\beta-2}. \tag{398}$$

Hence:

$$\begin{aligned} dy &= \left(\mu s y'(s) + \frac{1}{2}\zeta^2 s^2 y''(s)\right) dt + \zeta s y'(s) d\omega \\ &= \left(\mu s A\beta s^{\beta-1} + \frac{1}{2}\zeta^2 s^2 A\beta(\beta-1)s^{\beta-2}\right) dt + \zeta s A\beta s^{\beta-1} d\omega \\ &= \left(\beta\mu A s^{\beta} + \frac{1}{2}\beta(\beta-1)\zeta^2 A s^{\beta}\right) dt + \beta\zeta A s^{\beta} d\omega \\ &= \left(\beta\mu + \frac{1}{2}\beta(\beta-1)\zeta^2\right) y dt + \beta\zeta y d\omega. \end{aligned} \tag{399}$$

Ito's lemma can thus be used to show that the power function of a geometric Brownian motion with drift μs and volatility ζs is a geometric Brownian motion with drift $(\beta\mu + \beta(\beta-1)\zeta^2/2)y$ and volatility $\beta\zeta y$.

15.4.4 Substraction of a constant

Let $s(t)$ be a geometric Brownian motion:

$$ds = \mu s dt + \zeta s d\omega, \qquad s > 0. \tag{400}$$

One may interpret $s(t)$ as the present discounted value (PDV) of an investment project and I as the cost of installation of the project. Now consider the function $y(s)$ representing the PDV of the project net of its installation cost:

$$y(s) = s - I, \quad y'(s) = 1, \quad y'(s) = 0. \tag{401}$$

Ito's lemma yields:

$$\begin{aligned} dy &= \left(y'(s)\mu s + \frac{1}{2} y''(s)\zeta^2 s^2 \right) dt + y'(s)\zeta s d\omega \\ &= \mu s dt + \zeta s d\omega \\ &= \mu(y + I)dt + \zeta(y + I)d\omega. \end{aligned} \tag{402}$$

Hence:

$$\frac{dy}{y} = \left(\mu + \frac{\mu I}{y} \right) dt + \left(\zeta + \frac{\zeta I}{y} \right) d\omega. \tag{403}$$

16 Continuous-time dynamic programming

References: Chang (2004).

16.1 Digression on the number e and the return on an asset in an infinitesimally small period

The number e is defined as:

$$\begin{aligned} e &= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\ &= \lim_{n\to\infty}\left(1+\frac{100\%}{n}\right)^n. \end{aligned} \tag{404}$$

The number e can thus be interpreted as the compound return on an asset that earns an interest of 100% per period and is continually reinvested during one period.

This definition can be used to express $e^{\rho t}$ as follows:

$$\begin{aligned} e^{\rho t} &= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{\rho t} \\ &= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\rho}\right]^{t} \\ &= \left[\lim_{n'\to\infty}\left(1+\frac{\rho}{n'}\right)^{n'}\right]^{t} \end{aligned} \tag{405}$$

where $n' = n\rho$ and use is made of the power rule for limits. The term $e^{\rho t}$ can thus be interpreted as the compound return on an asset that earns an interest of ρ per period an is continually reinvested during t periods.

It is intuitive that the net return on an asset that earns an interest of ρ during an infinitesimally small period is the same whether or not the asset is continually reinvested. Indeed, using L'Hôpital's rule, one can show this as follows:

$$\lim_{dt\to 0}\frac{e^{\rho dt}-1}{dt} = \lim_{dt\to 0}\frac{\rho e^{\rho dt}}{1} = \rho. \tag{406}$$

16.2 Optimization with one state variable and one control variable and discounted rewards (model class 1)

16.2.1 Optimization problem

The optimization problem is as follows:

- At each instant of time, an agent observes the state variable s, takes an action x and receives an instantaneous reward $f(s,x)$, which is discounted by a discount factor ρ.
- The state is governed by the following stochastic differential:

$$ds = g(s,x)dt + \sigma(s)d\omega. \tag{407}$$

- The agent's optimization problem may be stated as follows:

$$\max_x \mathrm{E}\left(\int_0^\infty e^{-\rho t} f(s,x)dt\right), \tag{408}$$

such that

$$ds = g(s,x)dt + \sigma(s)d\omega. \tag{409}$$

- The agent seeks an optimal policy $x = x(s)$, which stipulates what action should be taken in state s so as to maximize the expected stream of rewards over time.

16.2.2 Bellman equation

Let $V(s)$ be the value function, that is, the maximum expected stream of rewards over time, given the state s.

According to Bellman's principle of optimality, the value function must satisfy:

$$V(s) = \max_x \big(f(s,x)dt + e^{-\rho dt}\,\mathrm{E}(V(s+ds))\big) \tag{410}$$

Multiplying through with $e^{\rho dt}/dt$ and subtracting $V(s)/dt$ yields:

$$\frac{e^{\rho dt}-1}{dt}V(s) = \max_x \left\{ e^{\rho dt} f(s,x) + \mathrm{E}\left(\frac{V(s+ds)-V(s)}{dt}\right)\right\}. \tag{411}$$

The Bellman equation is thus:

$$\rho V(s) = \max_x \left\{ f(s,x) + \mathrm{E}\left(\frac{dV(s)}{dt}\right)\right\}. \tag{412}$$

Note that use has been made of the fact that, as shown in section 16.1, $(e^{\rho dt}-1)/dt = \rho$.

Ito's lemma provides the stochastic differential of the value function $V(s)$:

$$dV(s) = \left(g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss}\right)dt + \sigma(s)V_s d\omega. \tag{413}$$

Hence:

$$\mathrm{E}\left(\frac{dV(s)}{dt}\right) = g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss}. \tag{414}$$

Plugging equation (414) into equation (412), we obtain the Bellman equation in continuous time for an optimization problem with one state variable and one control variable and discounted rewards:

$$\rho V(s) = \max_x \left\{ f(s,x) + g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss} \right\}. \tag{415}$$

To see the intuition of the Bellman equation stated above, think of $V(s)$ as the value of a stream of income generated by an optimally managed asset over time. The owner of the asset has two options:

- either sell the asset and invest the proceeds, obtaining a return $\rho V(s)$;
- or hold it, realizing a dividend $f(s,x)dt$ and an expected capital appreciation $\mathrm{E}(dV(s)/dt)$.

Bellman's equation states that the returns are equal in both cases and that arbitrage is impossible.

16.3 Optimization with n state variables and k control variables and discounted rewards (model class 2)

16.3.1 Optimization problem

The optimization problem is as follows:

- At each instant of time, an agent observes the vector of state variables $s = (s_1, \ldots, s_n)'$, takes actions summarized in the vector $x = (x_1, \ldots, x_k)'$ and receives an instantaneous reward $f(s, x)$, which is discounted by a discount factor ρ.
- The state variables are governed by the following stochastic differentials:

$$ds_i = g_i(s, x)dt + \sigma_i(s)d\omega_i, \quad i = 1, \ldots, n. \tag{416}$$

- The agent's optimization problem may be stated as follows:

$$\max_x \mathrm{E}\left(\int_0^\infty e^{-\rho t} f(s, x)dt\right), \tag{417}$$

  such that

$$ds_i = g_i(s, x)dt + \sigma_i(s)d\omega_i, \quad i = 1, \ldots, n. \tag{418}$$

- The agent seeks an optimal policy $x = x(s)$, which stipulates what actions should be taken in state s so as to maximize the expected stream of rewards over time.

16.3.2 Bellman equation

Let $V(s)$ be the value function, that is, the maximum expected stream of rewards over time, given the state s.

According to Bellman's principle of optimality, the value function must satisfy:

$$V(s) = \max_x \left\{ f(s,x)dt + e^{-\rho dt} \operatorname{E}(V(s+ds)) \right\} \tag{419}$$

$$\Leftrightarrow \quad \rho V(s) = \max_x \left\{ f(s,x) + \operatorname{E}\left(\frac{dV(s)}{dt}\right) \right\} \quad (\text{as } dt \to 0). \tag{420}$$

Ito's lemma provides the stochastic differential of the value function $V(s)$:

$$dV(s) = \left(\sum_{i=1}^{n} g_i(s,x)V_{s_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(s)\sigma_j(s)\eta_{ij}V_{s_i s_j} \right) dt + \sum_{i=1}^{n} \sigma_i(s)V_{s_i} d\omega_i. \tag{421}$$

Note that Ito's lemma can be derived using a second-order Taylor approximation:

$$dV(s) = \sum_{i=1}^{n} V_{s_i} ds_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{s_i s_j} ds_i ds_j, \tag{422}$$

where

$$ds_i = g_i(s,x)dt + \sigma_i d\omega_i, \tag{423}$$

$$ds_i ds_j = \begin{cases} \sigma_i^2(s)dt & \text{if } i = j, \\ \sigma_i(s)\sigma_j(s)\eta_{ij}dt & \text{if } i \neq j. \end{cases} \tag{424}$$

Plugging equation (421) into equation (420), we obtain the Bellman equation in continuous time for an optimization problem with n state variables and k control variables and discounted rewards:

$$\rho V(s) = \max_x \left\{ f(s,x) + \sum_{i=1}^{n} g_i(s,x) V_{s_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(s)\sigma_j(s)\eta_{ij} V_{s_i s_j} \right\}. \tag{425}$$

16.4 Optimization with one state variable and one control variable with undiscounted rewards (model class 3)

16.4.1 Optimization problem

The optimization problem is as follows:

- At each instant of time, an agent observes the state variable s, takes an action x and receives an instantaneous reward $h(s,x)$.
- The state is governed by the following stochastic differential:

$$ds = g(s,x)dt + \sigma(s)d\omega. \tag{426}$$

- The agent's optimization problem may be stated as follows:

$$\max_x \mathrm{E}\left(\int_0^{\infty} h(s,x)dt\right), \tag{427}$$

  such that

$$ds = g(s,x)dt + \sigma(s)d\omega. \tag{428}$$

- The agent seeks an optimal policy $x = x(s)$, which stipulates what action should be taken in state s so as to maximize the expected stream of rewards over time.

16.4.2 Bellman equation

Let $V(s)$ be the value function, that is, the maximum expected stream of rewards over time, given the state s.

According to Bellman's principle of optimality, the value function must satisfy:

$$V(s) = \max_{x} \left\{ h(s,x)dt + \mathrm{E}(V(s+ds)) \right\}. \tag{429}$$

Ito's lemma provides the stochastic differential of the value function $V(s)$:

$$dV(s) = \left(g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss} \right) dt + \sigma(s)V_s d\omega. \tag{430}$$

Hence:

$$\mathrm{E}(V(s+ds)) = V(s) + \left(g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss} \right) dt. \tag{431}$$

Plugging equation (431) into equation (429), we obtain the Bellman equation in continuous time for an optimization problem with one state variable and one control variable:

$$V(s) = \max_x \left\{ h(s,x)dt + V(s) + \left(g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss} \right) dt \right\} \tag{432}$$

$$\Leftrightarrow \quad 0 = \max_x \left\{ h(s,x) + g(s,x)V_s + \frac{1}{2}\sigma^2(s)V_{ss} \right\}. \tag{433}$$

16.5 Optimization with n state variables and k control variables with undiscounted rewards (model class 4)

16.5.1 Optimization problem

The optimization problem is as follows:

- At each instant of time, an agent observes the vector of state variables $s = (s_1, \ldots, s_n)'$, takes actions summarized in the vector $x = (x_1, \ldots, x_k)$ and receives an instantaneous reward $h(s, x)$.
- The state variables are governed by the following stochastic differentials:
$$ds_i = g_i(s, x)dt + \sigma_i(s)d\omega_i, \quad i = 1, \ldots, n. \tag{434}$$
- The agent's optimization problem may be stated as follows:
$$\max_x \mathrm{E}\left(\int_0^\infty h(s, x)dt\right), \tag{435}$$
such that
$$ds_i = g_i(s, x)dt + \sigma_i(s)d\omega_i, \quad i = 1, \ldots, n. \tag{436}$$
- The agent seeks an optimal policy $x = x(s)$, which stipulates what actions should be taken in state s so as to maximize the expected stream of rewards over time.

16.5.2 Bellman equation

Let $V(s)$ be the value function, that is, the maximum expected stream of rewards over time, given the state s.

According to Bellman's principle of optimality, the value function must satisfy:

$$V(s) = \max_{x} \left\{ h(s,x)dt + \mathrm{E}\left(V(s+ds)\right)\right\} \quad (\text{as } dt \to 0). \tag{437}$$

Ito's lemma provides the stochastic differential of the value function $V(s)$:

$$dV(s) = \left(\sum_{i=1}^{n} g_i(s,x) V_{s_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(s)\sigma_j(s)\eta_{ij} V_{s_i s_j} \right) dt + \sum_{i=1}^{n} \sigma_i(s) V_{s_i} d\omega_i. \tag{438}$$

Hence:

$$\mathrm{E}(V(s+ds)) = V(s) + \left(\sum_{i=1}^{n} g_i(s,x) V_{s_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i(s)\sigma_j(s)\eta_{ij} V_{s_i s_j} \right) dt \tag{439}$$

Plugging equation (439) into equation (437), we obtain the Bellman equation in continuous time for an optimization problem with n state variables and k control variables and discounted rewards:

$$V(s) = \max_x \left\{ h(s,x)dt + V(s) + \left(\sum_{i=1}^{n} g_i(s,x)V_{s_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(s)\sigma_j(s)\eta_{ij}V_{s_i s_j} \right) dt \right\} \tag{440}$$

$$\Leftrightarrow \quad 0 = \max_x \left\{ h(s,x) + \sum_{i=1}^{n} g_i(s,x)V_{s_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i(s)\sigma_j(s)\eta_{ij}V_{s_i s_j} \right\}. \tag{441}$$

17 Examples of continuous-time dynamic programming

17.1 Consumption choice

Maximization problem:

$$\max_{c_1} \mathrm{E}\left(\int_0^\infty e^{-\rho t} u(c_1) dt\right), \tag{442}$$

where

$$da_0 = (\alpha a_0 - c_1)dt + \zeta a_0 d\omega, \tag{443}$$
$$u(c_1) = \ln(c_1). \tag{444}$$

Note that wealth, a_0, is the state variable and consumption, $c_1 dt$, the control variable.

Bellman equation:

$$\rho V(a_0) = \max_{c_1}\left\{u(c_1) + (\alpha a_0 - c_1)V'(a_0) + \frac{1}{2}\zeta^2 a_0^2 V''(a_0)\right\}. \tag{445}$$

Suppose we multiply a_0 and c_1 through with ϕ. The law of motion would not change. The objective function would become:

$$\mathrm{E}\left(\int_0^\infty e^{-\rho t}\ln(\phi c_1)dt\right) = A\ln(\phi) + \mathrm{E}\left(\int_0^\infty e^{-\rho t}\ln(c_1)dt\right). \tag{446}$$

Therefore:

$$V(\phi a_0) = A\ln(\phi) + V(a_0). \tag{447}$$

Since ϕ can take any value, we can set $\phi = 1/a_0$:

$$V(1) = B = -A\ln(a_0) + V(a_0) \quad \Leftrightarrow \quad V(a_0) = A\ln(a_0) + B. \tag{448}$$

So we know the form of the value function. Note the similarity of the function form of the utility function and the value function. The first two derivatives of the value function are $V'(a_0) = A/a_0$ and $V''(a_0) = -A/a_0^2$.

The first-order condition implies:

$$u'(c_1) = V'(a_0), \tag{449}$$

an intuitive result. It follows that $c_1 = a_0/A$.

The Bellman equation can now be written as:

$$\rho\left[A\ln(a_0)+B\right]=\ln(a_0)-\ln(A)+\alpha A-1-\frac{1}{2}\zeta^2 A. \tag{450}$$

The parameters A and B can now be found by equating coefficients. Note that $A=1/\rho$, so that the agent consumes a constant fraction of wealth (consumption smoothing):

$$c_1=\rho a_0. \tag{451}$$

17.2 Stochastic growth

Reference: Miranda.

Maximization problem:

$$\max_{c_1}\mathrm{E}\left(\int_0^\infty e^{-\rho t}u(c_1)dt\right), \tag{452}$$

where

$$dk_0=(yk_0-c_1)dt, \tag{453}$$

$$dy=(ay-b)dt+\zeta\sqrt{y}d\omega, \tag{454}$$

$$u(c_1)=\ln(c_1). \tag{455}$$

There is only one good which is either consumed or invested. The production is given by yk_0, where y measures the productivity of capital, k_0. Note that y evolves stochastically. The control variable is consumption, c_1.

The value function $V(k_0, y)$ depends on the two state variables, k_0 and y. The Bellman equation is:

$$\rho V(k_0, y) = \max_{c_1} \left\{ u(c_1) + (yk_0 - c_1)V_k + (ay - b)V_y + \frac{1}{2}\zeta^2 y V_{yy} \right\}. \tag{456}$$

Let us guess that the value function takes the following form:

$$V(k_0, y) = A\ln(k_0) + By + C. \tag{457}$$

If we can find values for A, B and C that satisfy the Bellman equation, we can confirm that the guess of the value function is correct.

From the first-order condition we can derive the optimal consumption rule (policy function):

$$u'(c_1) = V_k \quad \Leftrightarrow \quad c_1 = \frac{1}{A}k_0. \tag{458}$$

Substituting the value function and the policy function into the Bellman equation yields:

$$\rho(A\ln(k_0) + By + C) = \ln(k_0) - \ln(A) + Ay - 1 + B(ay - b). \tag{459}$$

One can now find A, B and C by equating the coefficients of $\ln(k_0)$ and y and solving the equation consisting of the remaining terms. Note that $A = 1/\rho$, so that the agent consumes a constant proportion of capital in the optimum:

$$c_1 = \rho k_0. \tag{460}$$

18 Portfolio diversification and a new rule for the current account

References: Kraay and Ventura (2000).

Remark.

Note that from now on we use a subindex 0 (for example, z_0) to indicate a stock variable, a subindex 1 (for example, z_1) to indicate its first time derivative and a subindex 2 to indicate its second time derivative:

$$z_1 = \frac{dz_0}{dt}, \qquad z_2 = \frac{dz_1}{dt}. \tag{461}$$

The notation implies that $z_1 dt$ denotes the flow corresponding to the accumulated stock z_0.

18.1 The model

Assume there are three assets in which a representative consumer-investor can invest his wealth a_0:

- domestic capital, $k_0 = wa_0$,
- foreign capital, $k_0^* = w^* a_0$,

- foreign loans, $a_0 - k_0 - k_0^* = (1 - w - w^*)a_0$.

The risk-free loans pay a constant interest rate, r. We assume that the risk and return properties of capital can be represented by yield-less assets whose prices follow geometric Brownian motions:

$$dP = \pi P dt + \zeta P d\omega, \tag{462}$$

$$dP^* = \pi^* P^* dt + \zeta^* P^* d\omega^*, \tag{463}$$

where ω and ω^* are Wiener processes with increments that are normally distributed with $\mathrm{E}(d\omega) = \mathrm{E}(d\omega^*) = 0$, $\mathrm{E}[(d\omega)^2] = \mathrm{E}[(d\omega^*)^2] = dt$ and $\mathrm{E}[(d\omega)(d\omega^*)] = \eta dt$.

It can be shown that the consumer's budget equation is:

$$a_1 dt = [(\pi - r)k_0 + (\pi^* - r)k_0^* + r a_0 - c_1]\, dt + k_0 \zeta d\omega + k_0^* \zeta^* d\omega^*. \tag{464}$$

The consumer's problem is:

$$\max_{c_1, k_0, k_0^*} E\left(\int_0^\infty e^{-\rho t} u(c_1) dt\right), \tag{465}$$

with initial wealth, $a_{0,0}$, given.

The Bellman equation is:

$$\rho V(a_0) = \max_{c_1,k_0,k_0^*}\Big\{u(c_1) + [\pi k_0 + \pi^* k_0^* + r(a_0 - k_0 - k_0^*) - c_1]\, V_{a_0} \\ + \left[\frac{1}{2}k_0^2\zeta^2 + k_0 k_0^* \zeta\zeta^*\eta + \frac{1}{2}(k_0^*)^2(\zeta^*)^2\right] V_{a_0 a_0}\Big\} \tag{466}$$

The optimal consumption and portfolio rules must satisfy the following first-order conditions:

$$u'(c_1) = V_{a_0}, \tag{467}$$
$$(\pi - r)V_{a_0} + [k_0\zeta^2 + k_0^*\zeta\zeta^*\eta]V_{a_0 a_0} = 0, \tag{468}$$
$$(\pi^* - r)V_{a_0} + [k_0^*(\zeta^*)^2 + k_0\zeta\zeta^*\eta]V_{a_0 a_0} = 0. \tag{469}$$

Equation (467) says that consumption is chosen in such a way that, in current values, the marginal utility of consumption equals the marginal utility of wealth. Equations (468) and (469) determine the optimal asset holdings:

$$k_0 = \frac{\pi - r}{\zeta^2}\left(-\frac{V_{a_0}}{V_{a_0 a_0}}\right) - \frac{\zeta\zeta^*\eta}{\zeta^2}k_0^*, \tag{470}$$
$$k_0^* = \frac{\pi^* - r}{(\zeta^*)^2}\left(-\frac{V_{a_0}}{V_{a_0 a_0}}\right) - \frac{\zeta\zeta^*\eta}{(\zeta^*)^2}k_0. \tag{471}$$

For example, the absolute holding of the domestic asset depends on its excess return, the risk tolerance (the reciprocal of absolute risk aversion) with respect to wealth and the need to diversify the portfolio to achieve an optimal combination of return and risk. Hence we obtain:

$$k_0 = \frac{1}{1-\eta^2}\left[\frac{\pi - r}{\zeta^2} - \frac{\zeta\zeta^*\eta(\pi^* - r)}{\zeta^2(\zeta^*)^2}\right]\left(-\frac{V_{a_0}}{V_{a_0a_0}}\right), \tag{472}$$

$$k_0^* = \frac{1}{1-\eta^2}\left[\frac{\pi^* - r}{(\zeta^*)^2} - \frac{\zeta\zeta^*\eta(\pi - r)}{\zeta^2(\zeta^*)^2}\right]\left(-\frac{V_{a_0}}{V_{a_0a_0}}\right). \tag{473}$$

Equations (472) and (473) can also be solved for the excess returns:

$$\pi - r = \frac{\zeta\zeta^*\eta}{(\zeta^*)^2}(\pi^* - r) + \zeta^2(1-\eta^2)\left(-\frac{V_{a_0a_0}}{V_{a_0}}\right)k_0. \tag{474}$$

Thus the domestic excess return depends on a term that is proportional to the product of the correlation of domestic and foreign returns and foreign excess returns as well as on a term that is proportional to the consumer's risk aversion.

18.2 Solving the model

Then according to equations (467), (472) and (473), the control variables c_1, k_0 and k_0^* are all functions of V_{a_0} and $V_{a_0 a_0}$. As a consequence, the Bellman equation can be written as a (highly nonlinear) second-order differential equation of the value function $V(a_0)$. Closed-form solutions do not exist except in a few special cases to which we turn now.

Suppose we specify an explicit utility function, $u(c_1)$. Here are examples of utility functions that permit explicit solutions of the Bellman equation:

$u(c_1)$	$V(a_0)$	Optimal c_1	Optimal k_0
$-\frac{1}{\gamma}e^{-\gamma c_1}$	$-\frac{A}{\gamma}e^{-r\gamma a_0}$	$ra_0 - \frac{1}{\gamma}\ln(rA)$	$\frac{1}{1-\eta^2}\left[\frac{\pi-r}{\zeta^2} - \frac{\zeta\zeta^*\eta(\pi^*-r)}{\zeta^2(\zeta^*)^2}\right]\frac{1}{r\gamma}$
$\frac{1}{1-\gamma}c_1^{1-\gamma}$	$\frac{A^{-\gamma}}{1-\gamma}a_0^{1-\gamma}$	Aa_0	$\frac{1}{1-\eta^2}\left[\frac{\pi-r}{\zeta^2} - \frac{\zeta\zeta^*\eta(\pi^*-r)}{\zeta^2(\zeta^*)^2}\right]\frac{a_0}{\gamma}$
$\ln(c_1)$	$\frac{1}{A}\ln(a_0) + B$	Aa_0	$\frac{1}{1-\eta^2}\left[\frac{\pi-r}{\zeta^2} - \frac{\zeta\zeta^*\eta(\pi^*-r)}{\zeta^2(\zeta^*)^2}\right]a_0$

The constants A and B can be found by plugging the optimal controls into the Bellman equations and equating coefficients (see for instance Chang, 2004).

Note that for the exponential utility function with constant absolute risk aversion, it is the optimal capital stock that is constant; for the isoelastic utility function with constant relative risk aversion (and the logarithmic one, which is a special case), it is the optimal share of capital in wealth that is constant. This observation provides some intuition for the terms ”absolute” and ”relative” risk aversion.

Let’s look at the solution of the Bellman equation for the logarithmic utility function $u(c_1) = \ln(c_1)$. We have:

$$V(a_0) = \frac{1}{A}\ln(a_0) + B, \quad V_{a_0} = \frac{1}{Aa_0}, \quad V_{a_0 a_0} = -\frac{1}{Aa_0^2}, \tag{475}$$

$$c_1 = Aa_0, \tag{476}$$

$$k_0 = \frac{1}{1-\eta^2}\left[\frac{\pi - r}{\zeta^2} - \frac{\zeta\zeta^*\eta(\pi^* - r)}{\zeta^2(\zeta^*)^2}\right] a_0, \tag{477}$$

$$k_0^* = \frac{1}{1-\eta^2}\left[\frac{\pi^* - r}{(\zeta^*)^2} - \frac{\zeta\zeta^*\eta(\pi - r)}{\zeta^2(\zeta^*)^2}\right] a_0. \tag{478}$$

The implicit function theorem states, loosely speaking, that if we have an equation $F(x, y) = c$ that defines y implicitly as a function of x, $y = f(x)$, then we obtain the derivative of y as follows:

$$\frac{dy}{dx} = -\frac{F_1(x, y)}{F_2(x, y)}, \tag{479}$$

where $F_i(x, y)$ denotes the partial derivative of $F(x, y)$ with respect to its ith argument.

We can use the implicit function theorem to determine how k depends on a. Let's rewrite equation (477) as follows:

$$F(a_0, k_0(a_0)) = \frac{k_0}{a_0} = \frac{1}{1-\eta^2}\left[\frac{\pi - r}{\zeta^2} - \frac{\zeta\zeta^*\eta(\pi^* - r)}{\zeta^2(\zeta^*)^2}\right]. \tag{480}$$

Then it follows from the implicit function theorem that:

$$\frac{dk_0}{da_0} = -\frac{F_1'(a_0, k_0)}{F_2'(a_0, k_0)} = -\frac{-\frac{k_0}{a^2}}{\frac{1}{a_0}} = \frac{k_0}{a_0}. \tag{481}$$

Similarly we obtain:

$$\frac{dk_0^*}{da_0} = \frac{k_0^*}{a_0}. \tag{482}$$

We can thus conclude that the marginal unit of wealth is invested in just the same manner as the average one and that increases in wealth do not affect the composition of the country's portfolio.

Remark.

The result that a_0, k_0 and k_0^* move in proportion is rather trivial and it might not seem necessary to apply the implicit function theorem to equations (477) and (478) to arrive at it. However, the implicit function theorem proves useful for modifications of the model.

18.3 Saving, investment and the current account

Let's now study the behaviour of the current account. Recall that the current account is by definition equal to domestic saving minus domestic investment, $z_1 dt = da_0 - dk_0$, or to the net rise in foreign assets (here comprising foreign capital and foreign loans).

Suppose that due to a positive productivity shock, $d\omega > 0$, saving da_0 rises. According to the traditional intertemporal approach to the current account studied in section 13, the additional saving is entirely invested abroad. The current account improves therefore by the same amount as da_0 since $dk_0 = 0$.

By contrast, the new rule for the current account proposed by Kraay and Ventura (2000) holds that the portfolio composition is maintained when saving increases and that part of the additional saving is invested at home. This leaves us with two possibilities

- If a country is a creditor country so that $a_0 - k_0 > 0$, or $k_0/a_0 < 1$, the country reacts to a positive productivity shock by running current account surplus:

$$z_1 dt = da_0 - dk_0 = d\omega - \frac{k_0}{a_0} d\omega > 0, \tag{483}$$

  where $z_1 dt$ is the current account.

- If, on the other hand, the country is a debtor country so that $a_0 - k_0 < 0$, or $k_0/a_0 > 1$, the country reacts to a positive productivity shock by running a current account deficit:

$$z_1 dt = da_0 - dk_0 = d\omega - \frac{k_0}{a_0} d\omega < 0. \tag{484}$$

18.4 Limitations

There are two important problems of the new rule to the current account.

- A positive income shock is modelled as a large positive value of $d\omega$. If the income shock is to be sustained during a longer period, say several years, $d\omega$ would have to be positive during a considerable time, an event whose probability tends to zero. In reality, countries experiencing economic booms often experience a rise of their return on capital relative to that in other countries. If this happens, however, it implies that investors shift their portfolios towards countries with high returns on capital, implying that those countries tend to run current account deficits.
- The other problem is that the new rule models physical investment using a model of financial investment. However, real investment is different from financial investment. Although both types of activities involve uncertainty, financial investment is reversible whereas real investment is not since it is normally associated with large sunk costs. To take account of this difference, it would be desirable to model the dynamics of aggregate real investment based on the principles of real options theory (Pindyck, 1991).

19 Investment based on real option theory

References: Dixit and Pindyck (1994), Miranda and Fackler (2002).

19.1 Digression on Euler's second-order differential equation

A differential equation of the following form is called Euler's differential equation:

$$s^2 y''(s) + asy'(s) + by = 0. \tag{485}$$

Here we will only consider a second-order Euler differential equation.

Let $s = e^t$, or $t = \ln(s)$. Then:

$$\frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = \frac{1}{s}\frac{dy}{dt}, \tag{486}$$

$$\begin{aligned}\frac{d^2y}{ds^2} &= \frac{d}{ds}\frac{dy}{ds}\\ &= \frac{d}{ds}\left(\frac{1}{s}\frac{dy}{dt}\right)\\ &= -\frac{1}{s^2}\frac{dy}{dt} + \frac{1}{s}\frac{d^2y}{dt^2}\frac{dt}{ds}\\ &= -\frac{1}{s^2}\frac{dy}{dt} + \frac{1}{s^2}\frac{d^2y}{dt^2}.\end{aligned} \tag{487}$$

To sum up, we have found the following:

$$y'(s) = \frac{1}{s}y'(t), \quad y''(s) = -\frac{1}{s^2}y'(t) + \frac{1}{s^2}y''(t). \tag{488}$$

Substituting the derivatives of $y(s)$ into Euler's differential equation yields:

$$s^2\left(-\frac{1}{s^2}y'(t) + \frac{1}{s^2}y''(t)\right) + as\frac{1}{s}y'(t) + by = 0 \tag{489}$$

$$\Leftrightarrow \quad y''(t) + (a-1)y'(t) + by = 0. \tag{490}$$

This is an ordinary second-order differential equation with constant coefficients.

If $(a-1)^2 > 4b$, then:

$$\begin{aligned} y(s) &= A_1 e^{r_1 t(s)} + A_2 e^{r_2 t(s)} \\ &= A_1 e^{r_1 \ln(s)} + A_2 e^{r_2 \ln(s)} \\ &= A_1 s^{r_1} + A_2 s^{r_2}, \end{aligned} \tag{491}$$

where

$$r_{1,2} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}. \tag{492}$$

The non-homogeneous equation is:

$$s^2 y''(s) + asy'(s) + by = cs + d. \tag{493}$$

Consider now the following particular solution:

$$y(s) = K_1 + K_2 s, \quad y'(s) = K_2, \quad y''(s) = 0. \tag{494}$$

Substituting the function $y(s)$ and its derivatives in the second-order Euler differential equation yields:

$$asK_2 + b(K_1 + K_2 s) = cs + d \tag{495}$$

$$\Rightarrow \quad K_1 = \frac{d}{b}, \quad K_2 = \frac{c}{a+b} \tag{496}$$

$$\Rightarrow \quad y(s) = K_1 + K_2 s = \frac{d}{b} + \frac{c}{a+b}s. \tag{497}$$

Thus the general solution is:

$$y(s) = A_1 s^{r_1} + A_2 s^{r_2} + \frac{d}{b} + \frac{c}{a+b}s. \tag{498}$$

19.2 Digression on the present discounted value of a profit flow

Suppose a real asset produces a profit flow governed by the following differential equation:

$$dx = \alpha x dt. \tag{499}$$

The present discounted value (PDV) of this profit flow is:

$$\begin{aligned} s_t &= \int_t^\infty x_\tau e^{-r(\tau-t)} d\tau \\ &= \int_t^\infty x_t e^{\alpha(\tau-t)} e^{-r(\tau-t)} d\tau \\ &= \left[\frac{1}{\alpha - r} x_t e^{(\alpha-r)(\tau-t)} \right]_t^\infty \\ &= \frac{x_t}{r-\alpha}. \end{aligned} \tag{500}$$

Note that a similar result can be obtained in discrete time:

$$\sum_{i=1}^\infty \frac{1}{(1+r)^i} (1+\alpha)^i x_t = \frac{1+\alpha}{1+r} \frac{1}{1-\frac{1+\alpha}{1+r}} x_t = \frac{1+\alpha}{r-\alpha} x_t = \frac{x_{t+1}}{r-\alpha}. \tag{501}$$

From Ito's lemma, we have:

$$ds = \alpha x \frac{1}{r - \alpha} dt = \alpha s dt. \tag{502}$$

Hence under the made assumptions x_t and s_t have the same percentage drift.

19.3 Arbitrage pricing theorem

Reference: Miranda.

Consider **two assets whose values** $v(s,t)$ **and** $w(s,t)$ **are functions of an underlying process** $s(s,t)$ that follows an Ito process:

$$ds = g(s,t)dt + \sigma(s,t)d\omega. \tag{503}$$

From Ito's lemma, it follows that:

$$dv = g_v dt + \sigma_v d\omega, \tag{504}$$
$$dw = g_w dt + \sigma_w d\omega. \tag{505}$$

Let us now build a **risk-less portfolio** π consisting of 1 unit of v and h units of w:

$$\pi = v + hw. \tag{506}$$

From Ito's lemma, we know that:

$$d\pi = dv + hdw = (g_v + hg_w)dt + (\sigma_v + h\sigma_w)d\omega. \tag{507}$$

To ensure that π is indeed risk-less, we must choose h as follows:

$$h = -\frac{\sigma_v}{\sigma_w}. \tag{508}$$

Now we can make use of a **no-arbitrage condition.** Since portfolio π is risk-less, its rate of return must be equal to the risk-free interest rate r, for otherwise it would be possible to make unbounded profits without any risk:

$$d\pi = dv - \frac{\sigma_v}{\sigma_w}(dw + \delta w dt) = r\pi dt. \tag{509}$$

Here δdt is the cash payment that has to be made to go short in w (that is, the fee for lending one unit of w during a period of length dt).

By dividing the last equation by dt, one gets:

$$g_v - \frac{\sigma_v}{\sigma_w}(g_w + \delta w) = r\left(v - \frac{\sigma_v}{\sigma_w}w\right), \tag{510}$$

which after rearranging yields what is called the **fundamental arbitrage condition:**

$$\frac{g_v - rv}{\sigma_v} = \frac{g_w - (r-\delta)w}{\sigma_w} \equiv \theta_s. \tag{511}$$

One can rewrite the fundamental arbitrage condition as follows:

$$g_v = rv + \theta_s \sigma_v, \tag{512}$$
$$g_w = (r - \delta)w + \theta_s \sigma_w. \tag{513}$$

These equations are referred to as the **single-factor arbitrage pricing theorem.** The single-factor arbitrage pricing theorem states that the expected return on any asset whose value is a function of s equals the risk-free interest rate plus a risk premium that is proportional to the asset's volatility. The proportionality factor θ is called the **market price of risk** for assets whose values depend on s.

19.4 Derivative pricing

Consider a **derivative asset** whose value y depends on the price s of another, **underlying asset.** Let s follow an Ito process:

$$ds = g(s,t)dt + \sigma(s,t)d\omega. \tag{514}$$

The derivative asset might, for example, be a call option to buy the underlying asset at a specified price at a future date.

From Ito's lemma, we can derive the stochastic differential of y:

$$dy = \left(y_t + gy_s + \frac{1}{2}\sigma^2 y_{ss}\right)dt + \sigma y_s d\omega. \tag{515}$$

Hence:

$$g_y = y_t + gy_s + \frac{1}{2}\sigma^2 y_{ss}. \tag{516}$$

$$\sigma_y = \sigma y_s. \tag{517}$$

Now consider the **risk-less portfolio** π with the derivative asset in long and the underlying asset in short position:

$$\pi = y + hs, \tag{518}$$

where

$$h = -\frac{\sigma_y}{\sigma} = -\frac{\sigma y_s}{\sigma} = -y_s. \tag{519}$$

It follows from the **fundamental arbitrage condition** that:

$$\frac{g_y - ry}{\sigma_y} = \frac{g - (r-\delta)s}{\sigma} \tag{520}$$

$$\Leftrightarrow \quad y_t + gy_s + \frac{1}{2}\sigma^2 y_{ss} - ry = [g - (r-\delta)s]y_s \tag{521}$$

$$\Leftrightarrow \quad ry = y_t + (r-\delta)sy_s + \frac{1}{2}\sigma^2 y_{ss}. \tag{522}$$

The last equation is the so-called **Black-Scholes differential equation.** It must be satisfied by any asset y that derives its value from the underlying asset s. This differential equation can be solved analytically or numerically once boundary conditions are specified.

When y does not depend on t, we can rewrite the fundamental arbitrage condition as follows:

$$ry(s) = (r-\delta)sy'(s) + \frac{1}{2}\sigma^2 y''(s). \tag{523}$$

19.4.1 The value of a real option

Suppose the present discounted value of an investment project follows a geometric Brownian motion:

$$ds = \alpha s dt + \zeta s d\omega, \tag{524}$$

where $\alpha = r - \delta > 0$.

The derivative asset has to satisfy the **Black-Scholes differential equation:**

$$ry(s) = (r - \delta)sy'(s) + \frac{1}{2}\zeta^2 s^2 y''(s). \tag{525}$$

This is a **second-order homogeneous Euler differential equation:**

$$\frac{1}{2}\zeta^2 s^2 y''(s) + (r - \delta)sy'(s) - ry(s) = 0 \tag{526}$$

$$\Leftrightarrow \quad s^2 y''(s) + asy'(s) + by(s) = 0, \tag{527}$$

where

$$a = \frac{2(r - \delta)}{\zeta^2}, \quad b = -\frac{2r}{\zeta^2}. \tag{528}$$

Suppose that $(a-1)^2 > 4b$. Then the **solution** is (for $s \leq s^*$):

$$y(s) = A_1 s^{\beta_1} + A_2 s^{\beta_2}, \tag{529}$$

where

$$\beta_{1,2} = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2}. \tag{530}$$

Hence:

$$\begin{aligned} \beta_1 &= \frac{1}{2} - \frac{r-\delta}{\zeta^2} + \sqrt{\left(\frac{r-\delta}{\zeta^2} - \frac{1}{2}\right)^2 + \frac{2r}{\zeta^2}} > 1, \\ \beta_2 &= \frac{1}{2} - \frac{r-\delta}{\zeta^2} - \sqrt{\left(\frac{r-\delta}{\zeta^2} - \frac{1}{2}\right)^2 + \frac{2r}{\zeta^2}} < 0. \end{aligned} \tag{531}$$

We still need to determine A_1, A_2 and the strike value of the project, s^*. This can be achieved with the three **boundary conditions:**

$$\begin{aligned} y(0) &= 0 \\ y(s^*) &= s^* - I, \\ y'(s^*) &= 1 \qquad \text{(smooth-pasting condition).} \end{aligned} \tag{532}$$

The first condition implies that $A_2 = 0$.

The second condition implies:

$$A_1 = \frac{s^* - I}{(s^*)^{\beta_1}}. \tag{533}$$

Finally, the third condition implies:

$$y'(s^*) = \beta_1 A_1 (s^*)^{\beta_1 - 1} = \beta_1 y(s^*)(s^*)^{-1} = \beta_1 (s^* - I)(s^*)^{-1} = 1 \tag{534}$$

$$\Leftrightarrow \quad s^* = \frac{\beta_1}{\beta_1 - 1} I. \tag{535}$$

Note that $s^* > I$, contradicting the investment rule that suggests investing as soon as the present discounted value of a project exceeds the investment cost.

The **final solution** is therefore:

$$y(s) = \begin{cases} A_1 s^{\beta_1} & \text{if } s \leq s^*, \\ s - I & \text{if } s > s^*. \end{cases} \tag{536}$$

19.4.2 Sensitivity to parameter changes

A particularly important result is the following:

$$s^* = \frac{\beta_1}{\beta_1 - 1} I \tag{537}$$

Since $\beta_1 > 1$, this says that the trigger value s^* for starting the project is above the installation cost I. Moreover, the lower is β_1, the higher will be the gap between s^* and I since:

$$\frac{d}{d\beta_1} \frac{\beta_1}{\beta_1 - 1} = \frac{-1}{(\beta_1 - 1)^2} < 0. \tag{538}$$

Now how does β_1 depend on the parameters ζ, r and δ? Note that when we substitute the solution $y(s) = A_1 s^{\beta_1}$ into the differential equation (526), we obtain:

$$\frac{1}{2}\zeta^2 s^2 A_1 \beta_1(\beta_1 - 1)s^{\beta_1 - 2} + (r - \delta)sA_1\beta_1 s^{\beta_1 - 1} - rA_1 s^{\beta_1} = 0 \tag{539}$$

$$\Leftrightarrow \quad Q = \frac{1}{2}\zeta^2 \beta_1(\beta_1 - 1) + (r - \delta)\beta_1 - r = 0. \tag{540}$$

Note first that:

$$\frac{\partial Q}{\partial \beta_1} = \frac{1}{2}\zeta^2(2\beta_1 - 1) + (r - \delta) > 0 \tag{541}$$

Using the total differentials for each of the parameters, we may now determine how β_1 reacts to changes in ζ, r and δ:

$$\underbrace{\frac{\partial Q}{\partial \beta_1}}_{>0} \frac{\partial \beta_1}{\partial \zeta} + \underbrace{\frac{\partial Q}{\partial \zeta}}_{>0} = 0 \quad \Rightarrow \quad \frac{\partial \beta_1}{\partial \zeta} < 0, \tag{542}$$

$$\underbrace{\frac{\partial Q}{\partial \beta_1}}_{>0} \frac{\partial \beta_1}{\partial r} + \underbrace{\frac{\partial Q}{\partial r}}_{>0} = 0 \quad \Rightarrow \quad \frac{\partial \beta_1}{\partial r} < 0, \tag{543}$$

$$\underbrace{\frac{\partial Q}{\partial \beta_1}}_{>0} \frac{\partial \beta_1}{\partial \delta} + \underbrace{\frac{\partial Q}{\partial \delta}}_{<0} = 0 \quad \Rightarrow \quad \frac{\partial \beta_1}{\partial \delta} > 0. \tag{544}$$

Hence:

$$\zeta \uparrow, \quad r \uparrow, \quad \delta \downarrow \;\; \Rightarrow \quad \beta_1 \downarrow, \quad s^* \uparrow, \quad \frac{dy(s)}{y(s)} \downarrow . \tag{545}$$

Note also:

$$I \uparrow \;\; \Rightarrow \quad s^* \uparrow . \tag{546}$$

The implicit function theorem also permits us to calculate the derivatives directly:

$$\frac{\partial \beta_1}{\partial \zeta} = -\frac{\beta_1(\beta_1 - 1)}{\zeta^2(2\beta_1 - 1) + (r - \delta)}, \tag{547}$$

$$\frac{\partial \beta_1}{\partial r} = -\frac{2(\beta_1 - 1)}{\zeta^2(2\beta_1 - 1) + (r - \delta)}, \tag{548}$$

$$\frac{\partial \beta_1}{\partial \delta} = \frac{2\beta_1}{\zeta^2(2\beta_1 - 1) + (r - \delta)}, \tag{549}$$

$$\frac{\partial \beta_1}{\partial \alpha} = \frac{2}{\zeta^2(2\beta_1 - 1) + (r - \delta)}. \tag{550}$$

19.4.3 Neoclassical theory of investment

Suppose a real asset produces a profit flow governed by the following stochastic differential equation:

$$dx = \alpha x dt + \zeta x d\omega. \tag{551}$$

Let s_t be the present discounted value of this profit flow. As we have seen, it holds that:

$$s_t = \frac{x_t}{r - \alpha} \quad \Leftrightarrow \quad x_t = (r - \alpha)s_t. \tag{552}$$

According to the neoclassical theory of investment, a firm invests if x_t is greater or equal to $(r - \alpha)I$:

$$s_t^* = I \quad \Leftrightarrow \quad x_t^* = (r - \alpha)I. \tag{553}$$

The ratio of the expected present value of profits that would flow from an investment project to the cost of installation, s/I, is also known as Tobin's q. Tobin's q is used as a threshold that justifies investment. Note that q equals 1 in neoclassical investment theory but is greater, and possibly much greater, than 1 in real options theory:

$$q = \frac{\beta_1}{\beta_1 - 1} > 1. \tag{554}$$

19.5 Dynamic programming

Net present value of a real asset:

$$ds = \alpha s dt + \zeta s d\omega. \tag{555}$$

Bellman equation:

$$\rho y(s) = \mathrm{E}(dy) = \alpha s y'(s) + \frac{1}{2}\zeta^2 s^2 y''(s) \tag{556}$$

After rearranging, we have:

$$\frac{1}{2}\zeta^2 s^2 y''(s) + \alpha s y'(s) - \rho y(s) = 0. \tag{557}$$

Note that this equation is almost identical to equation (526) that was based on derivative pricing theory. The only difference is that the risk-free interest rate r is replaced by the discount rate ρ. Since the same boundary conditions apply, the solution can be found analogously.

20 Capital accumulation under uncertainty

Reference: Bertola and Caballero (1994).

Consider the following production function:

$$\tilde{y}_1 = Z_0 k_0^{\alpha}, \qquad 0 < \alpha < 1, \tag{558}$$

where the business conditions index, Z_0, follows a geometric Brownian motion process:

$$dZ_0 = \mu Z_0 dt + \zeta Z_0 d\omega. \tag{559}$$

For simplicity, let the price of capital, P_0, be constant and equal to one.

The business conditions index, Z_0, is supposed to depend positively on the demand for the country's output and on productivity and negatively on the cost of factors other than capital.

Capital is accumulated according to the following continuous-time stochastic differential equation:

$$dk_0 = (-\delta k_0 + k_1)dt, \tag{560}$$

where δ is the depreciation rate and $k_1 dt$ gross investment.

Now we consider two scenarios, one where investment is reversible (k_1 unconstrained) and one where it is irreversible ($k_1 \geq 0$).

20.1 Reversible investment

The value function is:

$$V(k_0, Z_0) = \max_{k_1} E\left(\int_0^\infty e^{-rt}(Z_0 k_0^\alpha - P_0 k_1)dt\right). \tag{561}$$

The corresponding Bellman equation is:

$$rV = \max_{k_1}\left\{Z_0 k_0^\alpha - P_0 k_1 + (-\delta k_0 + k_1)V_{k_0} + \mu Z_0 V_{Z_0} + \frac{1}{2}\zeta^2 Z_0^2 V_{Z_0 Z_0}\right\}. \tag{562}$$

Optimal reversible investment must thus satisfy:

$$V_{k_0} = P_0 = 1. \tag{563}$$

Let $v(k_0, Z_0) = V_{k_0}(k_0, Z_0)$. Thus from the previous equation we know that $v(\cdot) = 1$. Then if one differentiates the Bellman equation term by term with respect to k_0, one obtains:

$$rv(\cdot) = \alpha Z_0 k_0^{\alpha-1} - \delta v(\cdot) + \mu Z_0 v_{Z_0}(\cdot) + \frac{1}{2}\zeta^2 Z_0^2 v_{Z_0 Z_0}(\cdot), \tag{564}$$

$$\frac{\partial \tilde{y}_1}{\partial k_0} = \alpha Z_0 k_0^{\alpha-1} = r + \delta, \tag{565}$$

where $r + \delta$ is the so-called user cost of capital.

The optimal capital stock if investment is reversible is therefore:

$$k_0^r = \left(\frac{r+\delta}{\alpha Z_0}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha Z_0}{r+\delta}\right)^{\frac{1}{1-\alpha}}. \tag{566}$$

20.2 Irreversible investment

The value function in the case of irreversible investment is the same as before:

$$V(k_0, Z_0) = \max_{k_1} E\left(\int_0^\infty e^{-rt}(Z_0 k_0^\alpha - P_0 k_1)dt\right). \tag{567}$$

The Bellman equation remains also unchanged:

$$rV = \max_{k_1}\left\{Z_0 k_0^\alpha - P_0 k_1 + (-\delta k_0 + k_1)V_{k_0} + \mu Z_0 V_{Z_0} + \frac{1}{2}\zeta^2 Z_0^2 V_{Z_0 Z_0}\right\}. \tag{568}$$

However, optimal *irreversible* investment should satisfy the so-called complementary slackness conditions:

$$v(\cdot) \leq P_0 = 1 \qquad \text{for all } t, \tag{569}$$

$$v(\cdot) = P_0 = 1 \qquad \text{for all } t \text{ where } k_1 > 0. \tag{570}$$

The intuition for the complementary slackness conditions is straightforward:

- The production function exhibits diminishing returns on capital.
- Without restrictions on investment, one would choose investment up to the point where the marginal increase of the value function equals the price of capital.
- If, however, investment is restricted to be nonnegative, investment is carried out until the marginal increase of the value function equals the price of capital. Yet when there has been excessive investment in the past, there will be no new investment and the marginal increase of the value function will be equal or below the purchasing price of capital.

Hence the value function can still be expressed in terms of the marginal product of capital function, $v(\cdot)$, but this time it cannot be simplified so easily:

$$rv(\cdot) = \alpha Z_0 k_0^{\alpha-1} - \delta v(\cdot) + \mu Z_0 v_{Z_0}(\cdot) + \frac{1}{2}\zeta^2 Z_0^2 v_{Z_0 Z_0}(\cdot). \tag{571}$$

A particular solution to this second-order ordinary differential equation is:

$$v^p(\cdot) = \frac{\alpha Z_0 k_0^{\alpha-1}}{r + \alpha\delta - \mu}. \tag{572}$$

The homogeneous solutions are of the form:

$$v^h(\cdot) = \left(\alpha Z_0 k_0^{\alpha-1}\right)^B P_0^{1-B}, \tag{573}$$

where B is a root, or solution, to the characteristic equation:

$$\frac{1}{2}\zeta^2 B^2 + \left(\mu + (1-\alpha)\delta - \frac{1}{2}\zeta^2\right) B - (r+\delta) = 0. \tag{574}$$

The roots B_1 and B_2 can be found using the formula for solving quadratic equations:

$$B_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{575}$$

where

$$a = \frac{1}{2}\zeta^2, \tag{576}$$

$$b = \mu + (1-\alpha)\delta - \frac{1}{2}\zeta^2, \tag{577}$$

$$c = -(r+\delta). \tag{578}$$

Here we consider only the positive root of the characteristic equation to allow for the possibility that $Z_0 = 0$. Thus we can denote B_1 simply as B.

The solution to equation (571) is therefore:

$$v(k_0, Z_0) = \frac{\alpha Z_0 k_0^{\alpha-1}}{r + \alpha\delta - \mu} + C\left(\alpha Z_0 k_0^{\alpha-1}\right)^B P_0^{1-B}, \tag{579}$$

where C is a constant to be determined based on the initial conditions.

An initial condition can be found by noting that for the desired capital stock it must hold that:

$$v(k_0^i, Z_0) = P_0 = 1, \tag{580}$$

$$v_{P_0}(k_0^i, Z_0) = 1. \tag{581}$$

From the second equation, we can deduce that:

$$C = \frac{1}{1-B}\left(\alpha Z_0 (k_0^i)^{\alpha-1}\right)^{-B} P_0^B, \tag{582}$$

$$v(k_0, Z_0) = \frac{\alpha Z_0 k_0^{\alpha-1}}{r+\alpha\delta-\mu} + \frac{1}{1-B}\left(\frac{\alpha Z_0 k_0^{\alpha-1}}{\alpha Z_0 (k_0^i)^{\alpha-1}}\right)^B P_0. \tag{583}$$

Combining this result with the fact that $v(k_0^i, Z_0) = P_0$, one finds that:

$$v(k_0^i, Z_0) = \frac{\alpha Z_0 (k_0^i)^{\alpha-1}}{r+\alpha\delta-\mu} + \frac{1}{1-B} v(k_0^i, Z_0) \tag{584}$$

$$\Leftrightarrow \quad \alpha Z_0 (k_0^i)^{\alpha-1} = \frac{B}{B-1}(r+\alpha\delta-\mu), \tag{585}$$

From the characteristic equation, it can be deduced that:

$$B(B-1)\zeta^2 + (B-1)\delta + B(\mu-\alpha\delta) - r = 0 \tag{586}$$

$$\Leftrightarrow \quad B(B-1)\zeta^2 + (B-1)\delta + B(\mu-\alpha\delta-r) + (B-1)r = 0 \tag{587}$$

$$\Leftrightarrow \quad \frac{B}{B-1} = \frac{r+\delta+B\zeta^2}{r+\alpha\delta-\mu}. \tag{588}$$

Hence the marginal productivity is given by:

$$\frac{\partial \tilde{y}(k_0^i, Z_0)}{\partial k_0} = \alpha Z_0 (k_0^i)^{\alpha-1} = r + \delta + B\zeta^2. \tag{589}$$

Finally, the optimal, though not always attainable, capital stock when investment is irreversible, k_0^i, is given by the following equation:

$$k_0^i = \left(\frac{r + \delta + B\zeta^2}{\alpha Z_0}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha Z_0}{r + \delta + B\zeta^2}\right)^{\frac{1}{1-\alpha}} < k_0^r. \tag{590}$$

20.3 Implications for the current account and currency market pressure

If investment is irreversible, we observe that an economic boom leads to a massive rise in gross investment. Yet suppose that expectations were too optimistic. Then there will be no corresponding ”negative” gross investment during the ensuing recession.

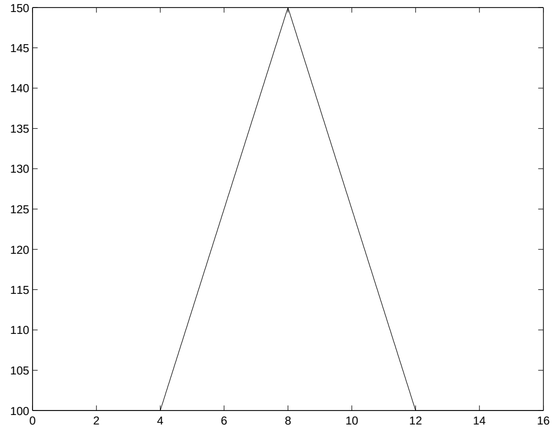
Recall that the current account is defined as saving minus real investment. As a result, one often observes a burgeoning current account deficit during a boom triggered by a sudden rise in the return on real capital, and an improvement of the current account during the recession. Overall, however, the international investment position, or cumulative current account balance, deteriorates during a boom-and bust cycle.

Currency market pressure is defined as the cumulative international cash flow of a country (cumulative current account balance plus net cumulative capital inflows) minus the net purchases of official reserves by the domestic central bank plus the net purchases of official reserves by the foreign central bank (see lecture notes on currency flows and currency crises). During a boom-and-bust cycle happens the following:

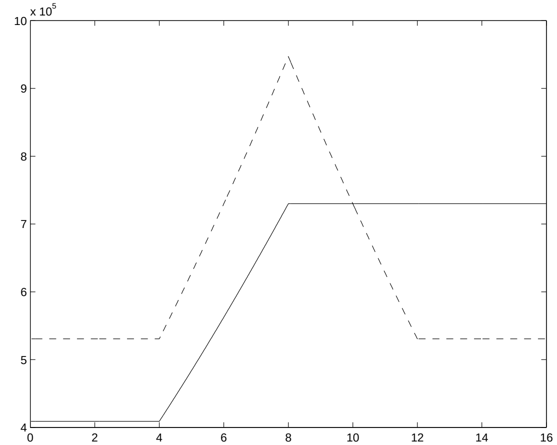
- Cumulative net capital inflows first rise and then fall back to more or less their initial level.
- The international investment position deteriorates.
- Currency market pressure, and thus the real exchange rate, first rises and then falls even below its initial level.

- If the domestic central bank wants to stabilize the currency, it can buy reserves and then sell them. When the central bank runs out of reserves, however, it will have to discontinue intervention and let the currency depreciate.

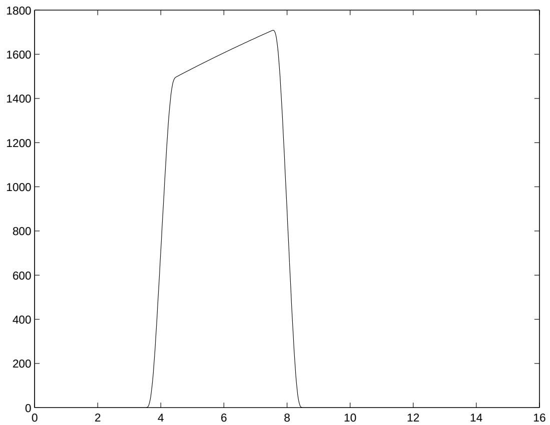
This is illustrated in figure 1. Here simple assumptions have been made, for example that $Z_{0,0} = 100$, $P_0 = 1$, $\alpha = 0.3$, $\mu = 0$ etc. Shocks have been set to zero for the simulation. Only the part of currency market pressure is calculated that depends on real investment (relevant for the current account) and net cumulative capital inflows.



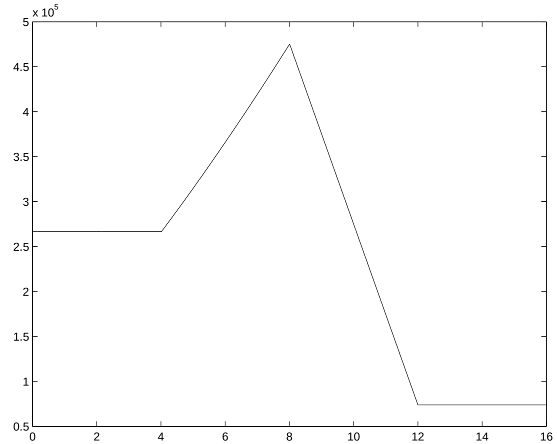
(a) Business conditions index



(b) Capital when investment is either reversible (dashed line) or irreversible (solid line)



(c) Irreversible investment



(d) Currency market pressure

Figure 1: **Capital accumulation under uncertainty.**

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