International macroeconomics
Additional material

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1 Mathematical concepts

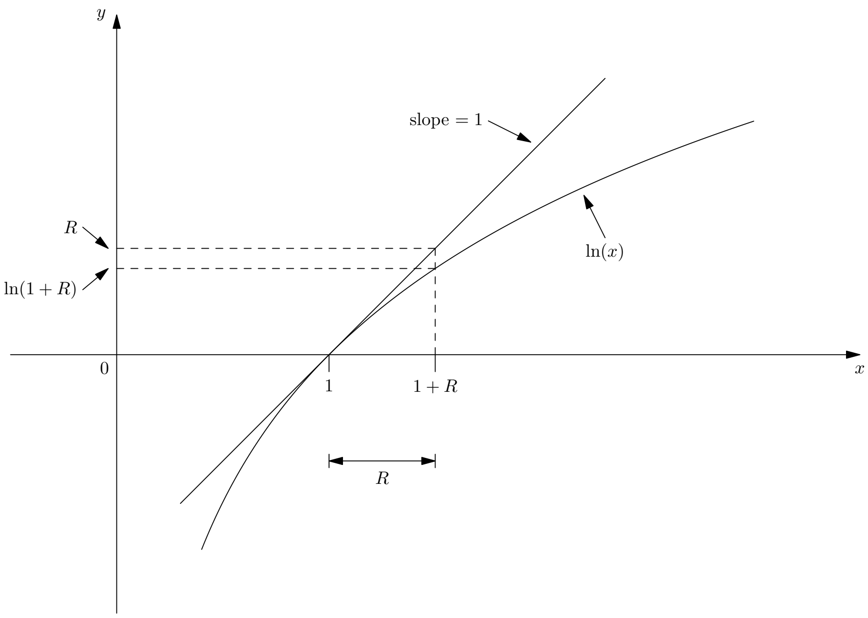

Figure 1: Log approximation.

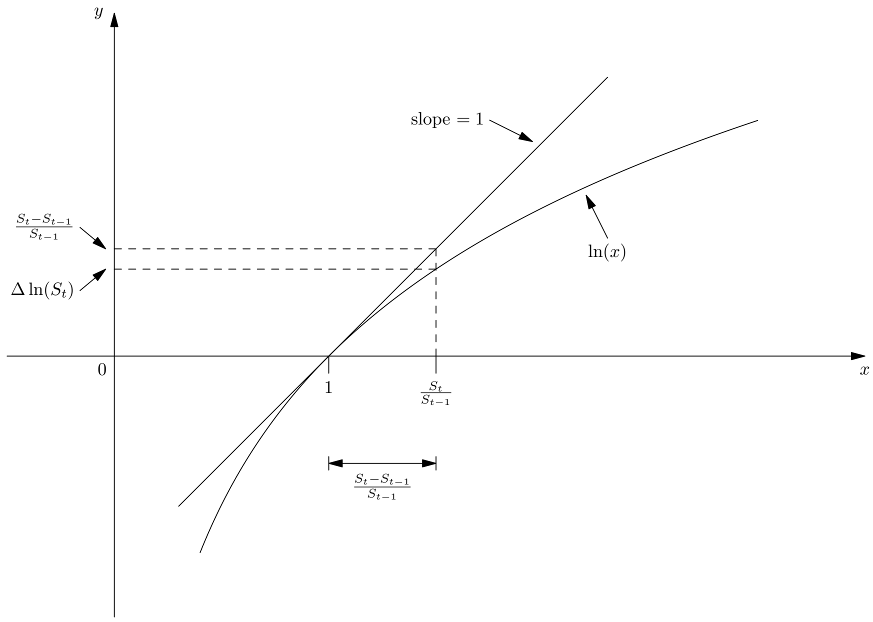

Figure 2: Log differencing.

2 Balance of payments

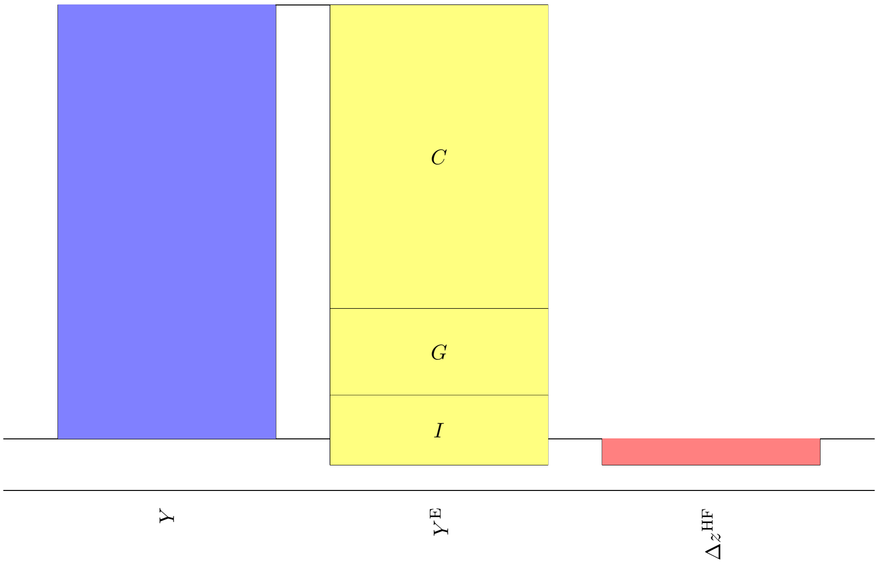

Figure 3: The current account as gross national disposable income minus gross national expenditure.

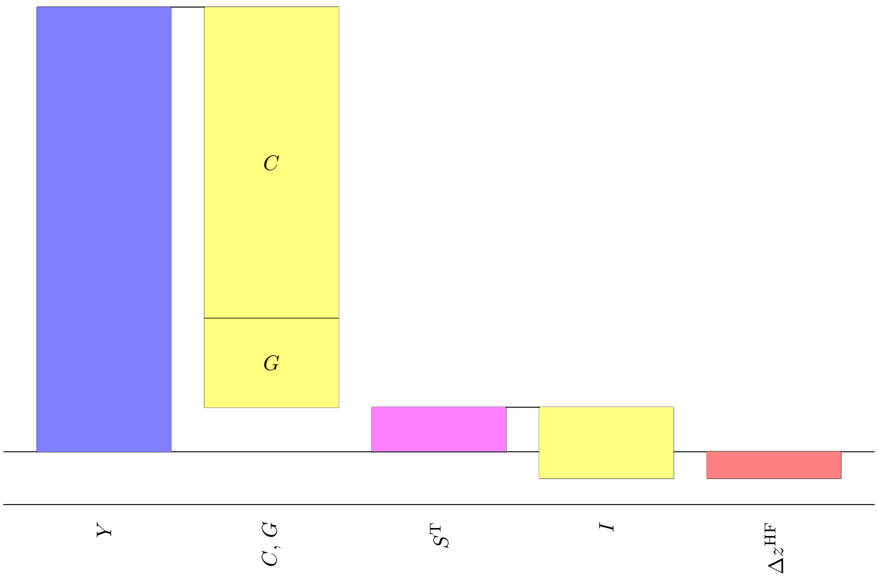

Figure 4: The current account as saving minus investment.

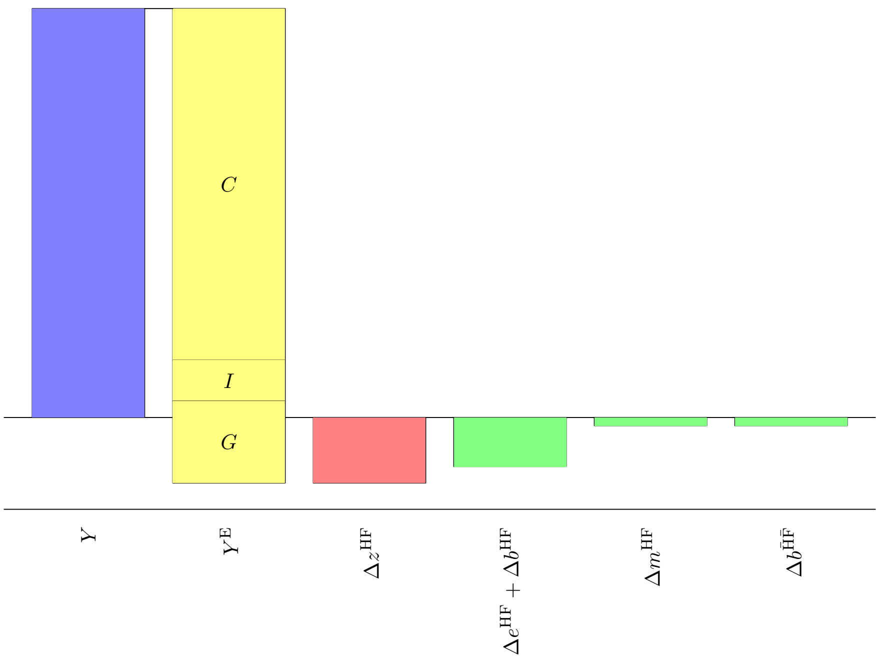

Figure 5: Current account deficit associated with capital inflows, money outflows and reserve losses.

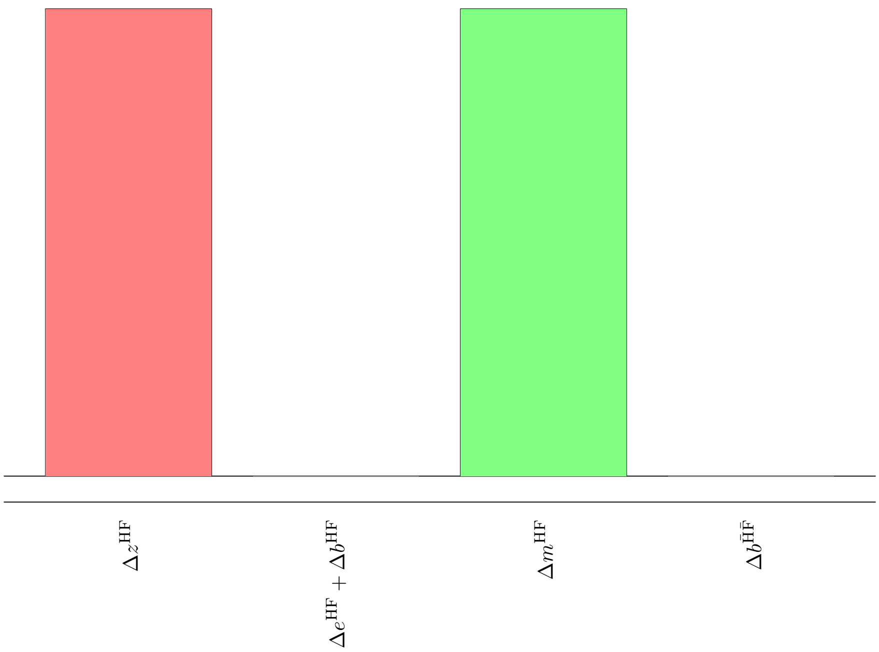

Figure 6: Current account surplus associated with money inflows.

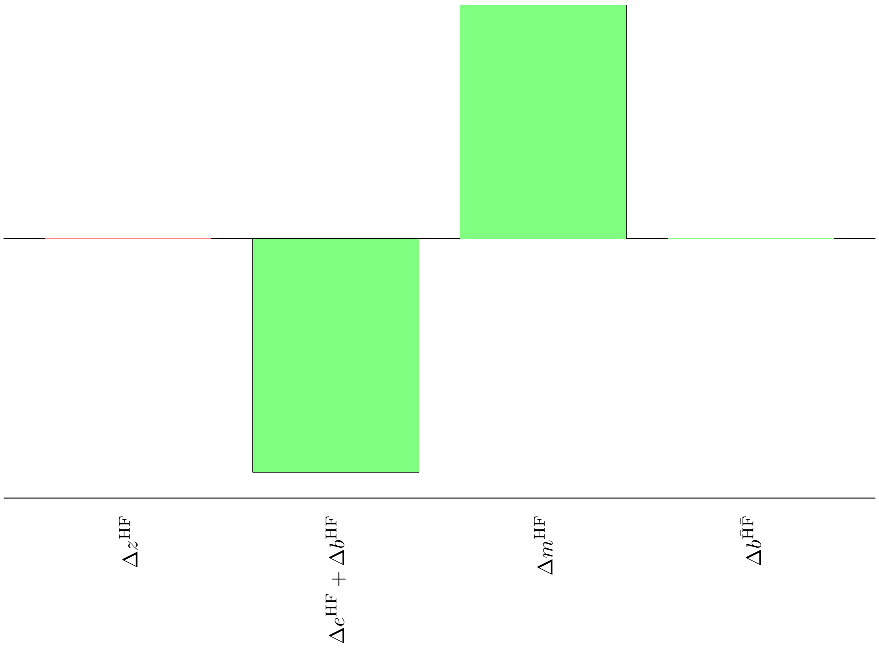

Figure 7: Capital inflows associated with money inflows.

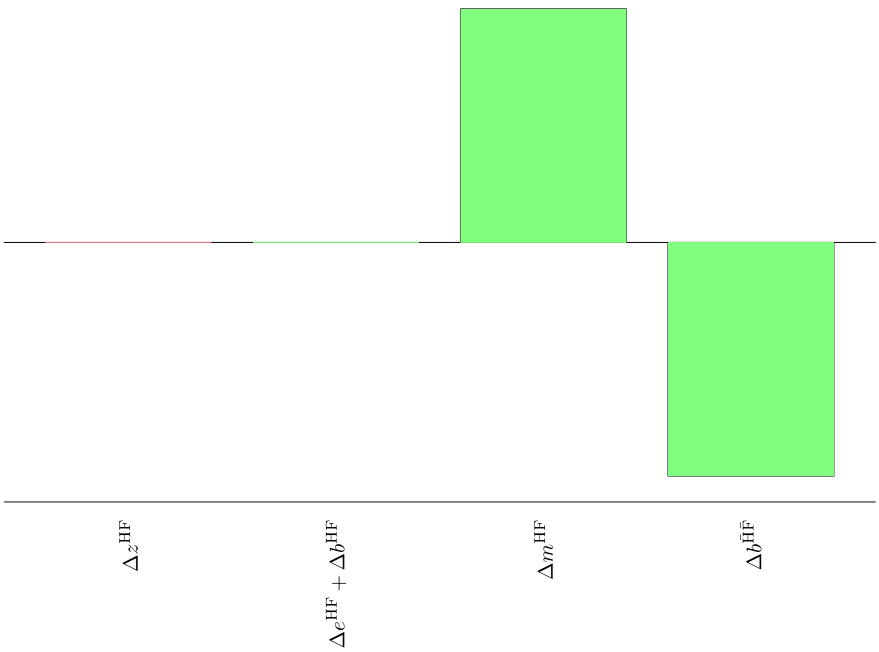

Figure 8: Sales of official reserves associated with money inflows.

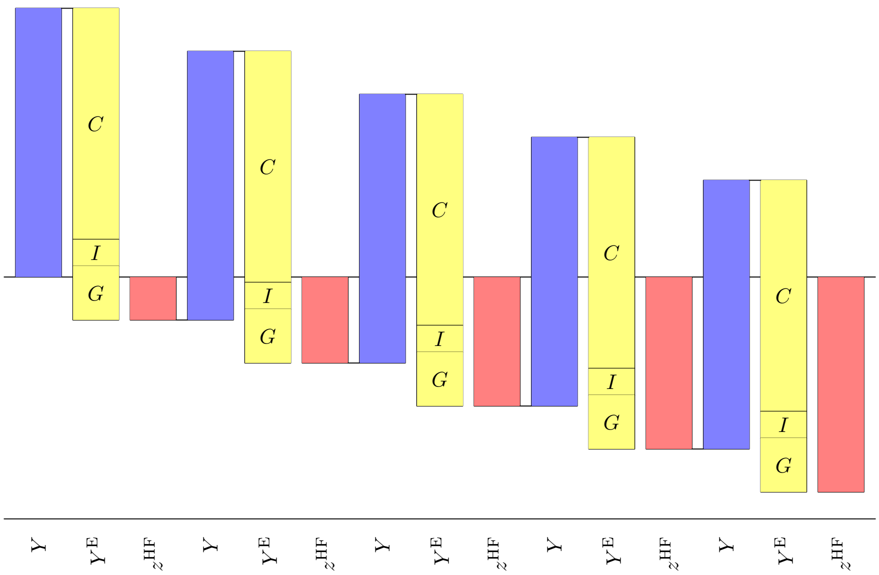

Figure 9: The current account and the international investment position.

3 Log-differencing

3.1 The natural logarithm

The number e is defined as follows:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828. \tag{1}$$

Suppose that $n = 1$. Then, the term of which the limit is taken is:

$$(1 + 1)^1 = 2. \tag{2}$$

This can be interpreted as an investment of one euro with a return of 100%.

Now suppose that $n = 2$. Now the term is:

$$(1 + 0.50)^2 = 2.25. \tag{3}$$

This can be interpreted as an investment of one euro that is reinvested once during the same period, but with a return of only 50%, or $1/2$, in each subperiod. The final return, also called compound return, is 125%.

We can continue to raise n in this way. For example, if $n = 5$, the term becomes:

$$(1 + 0.20)^5 = 2.48832. \tag{4}$$

This amounts to one euro being invested and reinvested five times during a given period, but with a return of only 20%, or $1/5$, in each subperiod. The compound return is 149%.

Now let's suppose we raise n to 100. Then we have:

$$(1 + 0.01)^{100} = 2.70481, \tag{5}$$

which is already very close to e. This can be interpreted as an investment of one euro that is invested and reinvested one hundred times, but with a return of only 1%, or $1/100$, in each subperiod. Now, the compound return is approximately 170%.

We see that when n is high, or when $1/n$ is low, we obtain:

$$\left(1 + \frac{1}{n}\right)^n \approx e \tag{6}$$

$$\Leftrightarrow \quad n \ln\left(1 + \frac{1}{n}\right) \approx \ln(e) = 1 \tag{7}$$

$$\Leftrightarrow \quad \ln\left(1 + \frac{1}{n}\right) \approx \frac{1}{n}. \tag{8}$$

If we set $a = 1/n$, we have:

$$\ln(1 + a) \approx a. \tag{9}$$

Actually, there is another way to derive the same result, which is based on a first-order Taylor series approximation:

$$f(x_0 + a) \approx f(x_0) + f'(x)|_{x=x_0}(x - x_0). \tag{10}$$

If we let $f(x) = \ln(x)$ and $x_0 = 1$, then we have:

$$\ln(1 + a) \approx \ln(1) + \left.\frac{d\ln(x)}{dx}\right|_{x=1} \times (1 + a - 1) = a, \tag{11}$$

since $\ln(1) = 0$ and $\ln(x)/dx = 1/x$.

3.2 Interpreting regressions with logarithmic variables

The approximation in equations (9) and (11) tells us that when a variable x is raised by a small percentage a, its natural logarithm rises by a. This is helpful, for instance when we want to interpret an estimated regression equation. Consider a consumption equation and suppose that C is consumption and Y income and that $c = \ln(C)$ and that $y = \ln(Y)$. Then, depending on how it is formulated, a regression of consumption on income can be interpreted as follows:

$\hat{C} = \hat{\alpha} + \hat{\beta}Y.$	If Y rises by 1 euro, C rises by $\hat{\beta}$ euros.
$\hat{c} = \hat{\alpha} + \hat{\beta}Y.$	If Y rises by 0.01 euros, C rises by $\hat{\beta}$ percent.
$\hat{C} = \hat{\alpha} + \hat{\beta}y.$	If Y rises by 1 percent, C rises by $0.01 \times \hat{\beta}$ euros.
$\hat{c} = \hat{\alpha} + \hat{\beta}y.$	If Y rises by 1 percent, C rises by $\hat{\beta}$ percent.

3.3 Log-differencing

Another very useful application of equations (9) and (11) is log-differencing. This is a method of calculating the growth rate of a variable by first applying the natural logarithm to it and then applying the difference operator.

Let $x_t = \ln(X_t)$ and denote the growth rate of X_t as $\hat{X}_t$.

$$\begin{aligned}
\Delta x_t &= x_t - x_{t-1} \\
&= \ln(X_t) - \ln(X_{t-1}) \\
&= \ln\left(\frac{X_t}{X_{t-1}}\right) \\
&= \ln\left(1 + \frac{X_t - X_{t-1}}{X_{t-1}}\right) \\
&\approx \frac{X_t - X_{t-1}}{X_{t-1}} \\
&= \hat{X}_t.
\end{aligned} \tag{12}$$

To take an example, let's look at the growth rate of real GDP:

1. Definition of real GDP:

$$\bar{Y}_t^{\text{P}} = \frac{Y_t^{\text{P}}}{P_t}. \tag{13}$$

2. Taking logarithms:

$$\bar{y}_t^{\text{P}} = y_t^{\text{P}} - p_t. \tag{14}$$

3. Applying the difference operator:

$$\begin{aligned} \Delta \bar{y}_t^{\text{P}} &= \Delta y_t^{\text{P}} - \Delta p_t \\ &= g_t^{\text{P}} - \pi_t. \end{aligned} \tag{15}$$

We see that the growth rate of real GDP is equal to the growth rate of GDP minus the inflation rate.

As another example, we may use log-differencing to derive the rate of appreciation of the real exchange rate:

1. Definition of the real exchange rate:

$$Q_t = \frac{S_t P_t^{\text{H}}}{P_t^{\text{F}}}. \tag{16}$$

2. Taking logarithms:

$$q_t = s_t + p_t^{\text{H}} - p_t^{\text{F}}. \tag{17}$$

3. Applying the difference operator:

$$\begin{aligned} \Delta q_t &= \Delta s_t + \Delta p_t^{\text{H}} - \Delta p_t^{\text{F}} \\ &= \Delta s_t + \pi_t^{\text{H}} - \pi_t^{\text{F}}. \end{aligned} \tag{18}$$

This shows us that the rate of real appreciation is equal to the rate of nominal appreciation plus the domestic inflation rate minus the foreign inflation rate.

4 Volatilities of national income components

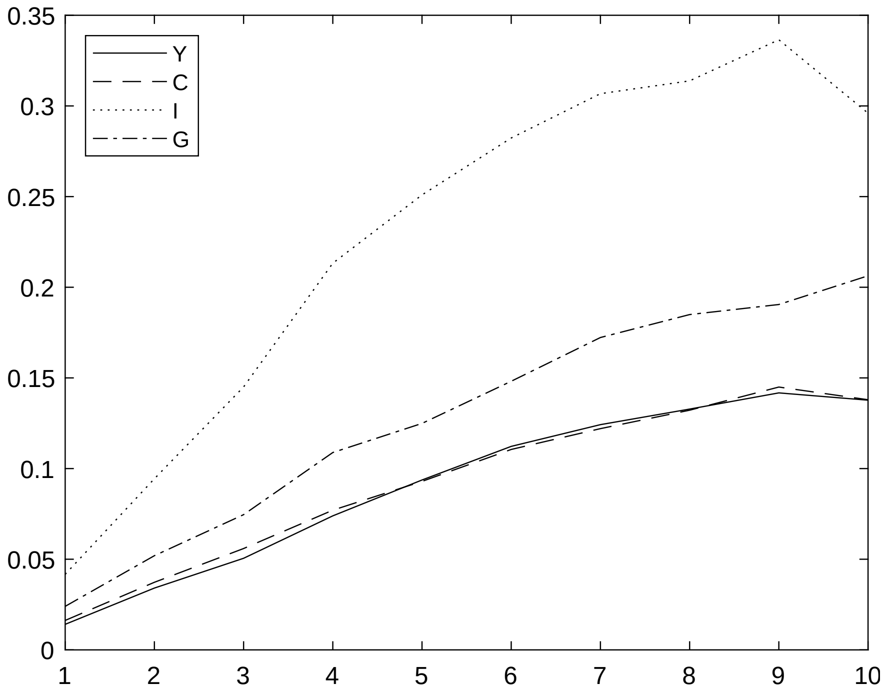

Figure 10: Volatilities of national income components at different horizons: national income (Y), private consumption (C), investment (I) and government spending (G). Horizons are measured in years. Source: International Financial Statistics (IMF), author's calculations.

The volatility of a given national income component, X, is measured as:

$$\begin{aligned}\mathrm{Var}(\Delta_h x_t) &= \mathrm{Var}(x_t - x_{t-h}) \\ &= \mathrm{Var}(\ln(X_t) - \ln(X_{t-h})) \\ &= \mathrm{Var}\left(\ln\left(\frac{X_t}{X_{t-h}}\right)\right) \\ &= \mathrm{Var}\left(\ln\left(1 + \frac{X_t - X_{t-h}}{X_{t-h}}\right)\right) \\ &\approx \mathrm{Var}\left(\frac{X_t - X_{t-h}}{X_{t-h}}\right).\end{aligned} \tag{19}$$

where h is the horizon.

Recall that $\ln(1 + a) \approx a$ if a is small.

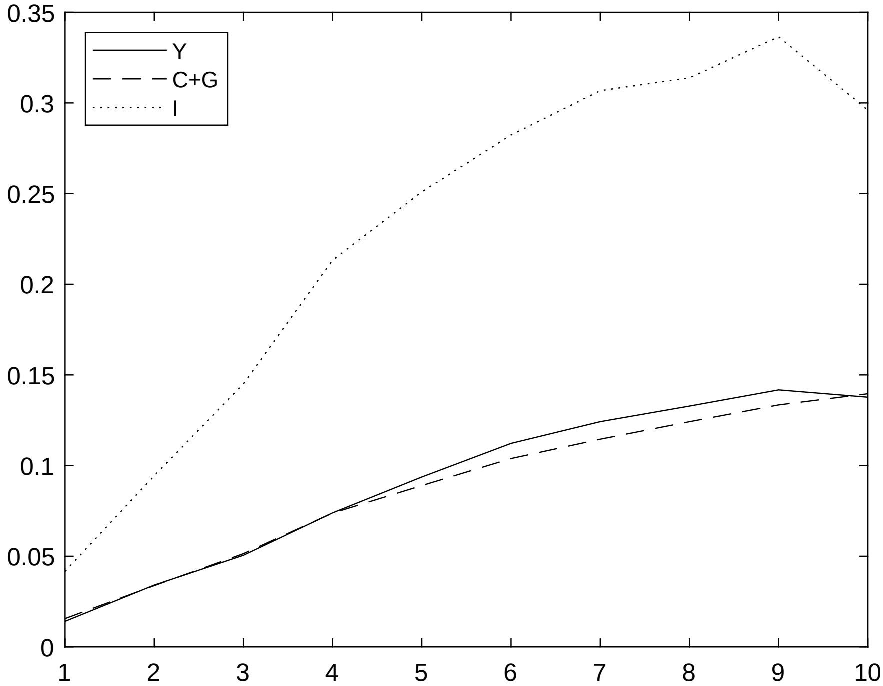

Figure 11: Volatilities of national income components at different horizons: national income (Y), private and public consumption ($C + G$) and investment (I). Horizons are measured in years. Source: International Financial Statistics (IMF), author's calculations.

5 Equity and debt finance

Suppose an enterprise or the government of a country wants to carry out a risky investment project and considers financing it by issuing shares or bonds to foreign investors.

The assumptions can be summarized as follows:

- Investment (cost of project): $I = 1000$
- Equity: E^{FH}
- Debt: B^{FH}
- Own money: $X = I - E^{\text{FH}} - B^{\text{FH}}$
- High return: $R_{I[+]} = +40\%$ (probability: $\mathrm{P}(I[+]) = 50\%$)
- Low return: $R_{I[-]} = -20\%$ (probability: $\mathrm{P}(I[-]) = 50\%$)
- Expected return: $\mathrm{E}(R_I) = \mathrm{P}(R_{I[+]})R_{I[+]} + \mathrm{P}(R_{I[-]})R_{I[-]}$
- Interest rate: $R_{B^{\text{FH}}} = 5\%$

5.1 Comparison of equity and bond finance

		Equity	Debt
Risk		Same as project risk	Higher than project risk (leverage effect)
Profit	Normal operation	Limited return	Leveraged return
	Project failure	Overseeable losses (limited to own invested money)	Leveraged losses (possibly exceeding own invested money)
Control	Normal operation	Shared with shareholders	Debtor
	Project failure	Shared with shareholders	Intervention by rescuers

Table 1: Comparison of equity and debt finance.

5.2 Funding through equity issuance

If the project is financed through equity issuance, the resulting return for both the project owner and equity holder is constant, as can be seen from table 2.

Equity	Project owner	Equity holder	Project owner	Equity holder	Project owner	Equity holder
	Average return (10%)		High return (40%)		Low return (-20%)	
0%	10%	—	40%	—	-20%	—
10%	10%	10%	40%	40%	-20%	-20%
20%	10%	10%	40%	40%	-20%	-20%
30%	10%	10%	40%	40%	-20%	-20%
40%	10%	10%	40%	40%	-20%	-20%
50%	10%	10%	40%	40%	-20%	-20%
60%	10%	10%	40%	40%	-20%	-20%
70%	10%	10%	40%	40%	-20%	-20%
80%	10%	10%	40%	40%	-20%	-20%
90%	10%	10%	40%	40%	-20%	-20%
100%	—	10%	—	40%	—	-20%

Table 2: Financing a project trough equity: returns for project owner and equity holder.

5.3 Funding through bond issuance

Debt	Leverage coefficient	Project owner	Lender	Project owner	Lender	Project owner	Lender
		Average return (10%)		High return (40%)		Low return (-20%)	
0%	1,00	10%	—	40%	—	-20%	—
10%	1,11	11%	5%	44%	5%	-23%	5%
20%	1,25	11%	5%	49%	5%	-26%	5%
30%	1,43	12%	5%	55%	5%	-31%	5%
40%	1,67	13%	5%	63%	5%	-37%	5%
50%	2,00	15%	5%	75%	5%	-45%	5%
60%	2,50	18%	5%	93%	5%	-58%	5%
70%	3,33	22%	5%	122%	5%	-78%	5%
80%	5,00	30%	5%	180%	5%	-120%	5%
90%	10,00	55%	5%	355%	5%	-245%	5%
100%	—	—	5%	—	5%	—	5%

Table 3: Equity versus debt financing: high investment return.

If the project is financed through bond issuance, due to the leverage effect the project owner can make very high profits when things go well, yet also end up with a pile of debt if they don't (table **??**). The return of the bond holder equals the coupon rate, no matter whether the project succeeds or fails.

Note that the leverage coefficient, which is an indicator of the riskiness of debt finance, is defined as follows:

$$\mathrm{LC} = \frac{I}{I - B^{\mathrm{FH}}}. \tag{20}$$

5.4 Combinded funding through equity and bond issuance

Equity and debt finance - two possibilities:

1. Equity investors do not share benefit and cost of debt funding.

   - Own profits:

   $$R_I(X + B^{\text{FH}}). \tag{21}$$

   - Payment to equity investors:

   $$R_I E^{\text{FH}}. \tag{22}$$

   - Payment to bond investors:

   $$R_{B^{\text{FH}}} B^{\text{FH}}. \tag{23}$$

   - Return on invested money, X:

   $$R_X = \frac{R_I(X + B^{\text{FH}}) - R_{B^{\text{FH}}} B^{\text{FH}}}{X}. \tag{24}$$

2. Equity investors share benefit and cost of debt funding.

   - Own profits:

   $$\frac{X}{X + E^{\text{FH}}} R_I I. \tag{25}$$

   - Payment to equity investors:

   $$\frac{E^{\text{FH}}}{X + E^{\text{FH}}} R_I I. \tag{26}$$

   - Payment to bond investors:

   $$\frac{X}{X + E^{\text{FH}}} R_{B^{\text{FH}}} B^{\text{FH}}. \tag{27}$$

   - Return on invested money, X:

   $$\begin{aligned} R_X &= \frac{\frac{X}{X+E^{\text{FH}}} R_I I - \frac{X}{X+E^{\text{FH}}} R_{B^{\text{FH}}} B^{\text{FH}}}{X} \\ &= \frac{R_I I - R_{B^{\text{FH}}} B^{\text{FH}}}{X + E^{\text{FH}}} \\ &= \text{LC} \times R_I - \frac{R_{B^{\text{FH}}} B^{\text{FH}}}{X + E^{\text{FH}}}, \end{aligned} \tag{28}$$

   where LC is the leverage coefficient:

   $$\text{LC} = \frac{I}{I - B^{\text{FH}}}. \tag{29}$$

Equity \ Debt	-	100	200	300	400	500	600	700	800	900	1.000
-	40,0%	43,9%	48,8%	55,0%	63,3%	75,0%	92,5%	121,7%	180,0%	355,0%	—
100	40,0%	44,4%	50,0%	57,5%	68,0%	83,8%	110,0%	162,5%	320,0%	—	
200	40,0%	45,0%	51,7%	61,0%	75,0%	98,3%	145,0%	285,0%	—		
300	40,0%	45,8%	54,0%	66,3%	86,7%	127,5%	250,0%	—			
400	40,0%	47,0%	57,5%	75,0%	110,0%	215,0%	—				
500	40,0%	48,8%	63,3%	92,5%	180,0%	—					
600	40,0%	51,7%	75,0%	145,0%	—						
700	40,0%	57,5%	110,0%	—							
800	40,0%	75,0%	—								
900	40,0%	—									
1.000	—										

Table 4: Equity versus debt financing: high investment return.

Equity \ Debt	-	100	200	300	400	500	600	700	800	900	1.000
-	-20,0%	-22,8%	-26,3%	-30,7%	-36,7%	-45,0%	-57,5%	-78,3%	-120,0%	-245,0%	—
100	-20,0%	-23,1%	-27,1%	-32,5%	-40,0%	-51,3%	-70,0%	-107,5%	-220,0%	—	
200	-20,0%	-23,6%	-28,3%	-35,0%	-45,0%	-61,7%	-95,0%	-195,0%	—		
300	-20,0%	-24,2%	-30,0%	-38,8%	-53,3%	-82,5%	-170,0%	—			
400	-20,0%	-25,0%	-32,5%	-45,0%	-70,0%	-145,0%	—				
500	-20,0%	-26,3%	-36,7%	-57,5%	-120,0%	—					
600	-20,0%	-28,3%	-45,0%	-95,0%	—						
700	-20,0%	-32,5%	-70,0%	—							
800	-20,0%	-45,0%	—								
900	-20,0%	—									
1.000	—										

Table 5: Equity versus debt financing: low investment return.

Equity \ Debt	-	100	200	300	400	500	600	700	800	900	1.000
-	10,0%	10,6%	11,3%	12,1%	13,3%	15,0%	17,5%	21,7%	30,0%	55,0%	—
100	10,0%	10,6%	11,4%	12,5%	14,0%	16,3%	20,0%	27,5%	50,0%	—	
200	10,0%	10,7%	11,7%	13,0%	15,0%	18,3%	25,0%	45,0%	—		
300	10,0%	10,8%	12,0%	13,8%	16,7%	22,5%	40,0%	—			
400	10,0%	11,0%	12,5%	15,0%	20,0%	35,0%	—				
500	10,0%	11,3%	13,3%	17,5%	30,0%	—					
600	10,0%	11,7%	15,0%	25,0%	—						
700	10,0%	12,5%	20,0%	—							
800	10,0%	15,0%	—								
900	10,0%	—									
1.000	—										

Table 6: Equity versus debt financing: average investment return.

6 Share prices

To derive the price of a share, we make use of the ”no-arbitrage” condition:

$$R^S_{t+1} = R_{t+1}. \tag{30}$$

This equation says that the return on a given share should be equal to the prevailing interest rate in the economy. For example, if $R^S_{t+1} > R_{t+1}$, the demand for the share and the share price will rise. As a result, the return on the share will fall, up to the point where equation (30) holds again.

We could add a risk premium to the right-hand side of equation (30) to take into account the fact that shares are more risky than bonds in general. For simplicity, however, in what follows we shall ignore the issue of risk and use equation (30) without risk premium.

Also recall the formula for calculating the sum of a geometric series:

$$1 + a + a^2 + \ldots = \frac{1}{1-a}. \tag{31}$$

Now we can use the formula for the return on a share, R^S_{t+1}, to compute the adequate price of the share, P^S_t.

$$R^S_{t+1} = \frac{P^S_{t+1} - P^S_t + D_{t+1}}{P^S_t} \tag{32}$$

$$\begin{aligned}
\Leftrightarrow \quad P^S_t &= \frac{P^S_{t+1} + D_{t+1}}{1 + R_{t+1}} \\
&= \frac{D_{t+1}}{1 + R_{t+1}} + \frac{P^S_{t+2} + D_{t+2}}{(1 + R_{t+1})(1 + R_{t+2})} \\
&= \frac{D_{t+1}}{1 + R_{t+1}} + \frac{D_{t+2}}{(1 + R_{t+1})(1 + R_{t+2})} + \frac{P^S_{t+3} + D_{t+3}}{(1 + R_{t+1})(1 + R_{t+2})(1 + R_{t+2})} \qquad (33) \\
&= \ldots \\
&= \lim_{h\to\infty} \sum_{i=1}^{h} \left(\frac{D_{t+i}}{\prod_{j=1}^{i}(1 + R_{t+j})} \right) + \frac{P_{t+h}}{\prod_{j=1}^{h}(1 + R_{t+j})}
\end{aligned}$$

Note that the denominator of the second term of the limit rises exponentially, so that the second term can be expected to fall to zero as the horizon, h, approaches infinity.

Now suppose that dividends grow at a constant rate g and that the interest rate is constant:

$$D_{t+h} = (1 + g)^{h-1} D_{t+1}, \tag{34}$$

$$R_t = R. \tag{35}$$

The above formula for the share price, P_t^S, then simplifies as follows:

$$\begin{aligned}
P_t^S &= \frac{D_{t+1}}{1+R} + \frac{(1+g)D_{t+1}}{(1+R)^2} + \frac{(1+g)^2 D_{t+1}}{(1+R)^3} + \dots \\
&= \frac{1}{1+R} \times \left(1 + \frac{(1+g)}{(1+R)} + \frac{(1+g)^2}{(1+R)^2} + \dots\right) \times D_{t+1} \\
&= \frac{1}{1+R} \times \left(\sum_{i=0}^{\infty} \left(\frac{1+g}{1+R}\right)^i\right) \times D_{t+1} \\
&= \frac{1}{1+R} \times \frac{1}{1-\frac{1+g}{1+R}} \times D_{t+1} \\
&= \frac{1}{1+R} \times \frac{1+R}{R-g} \times D_{t+1} \\
&= \frac{1}{R-g} \times D_{t+1}.
\end{aligned} \tag{36}$$

	0,00%	0,5%	1,0%	1,5%	2,0%	2,5%	3,0%	3,5%	4,0%	4,5%	5,0%	5,5%	6,0%
0,50%	200,0												
1,00%	100,0	200,0											
1,50%	66,7	100,0	200,0										
2,00%	50,0	66,7	100,0	200,0									
2,50%	40,0	50,0	66,7	100,0	200,0								
3,00%	33,3	40,0	50,0	66,7	100,0	200,0							
3,50%	28,6	33,3	40,0	50,0	66,7	100,0	200,0						
4,00%	25,0	28,6	33,3	40,0	50,0	66,7	100,0	200,0					
4,50%	22,2	25,0	28,6	33,3	40,0	50,0	66,7	100,0	200,0				
5,00%	20,0	22,2	25,0	28,6	33,3	40,0	50,0	66,7	100,0	200,0			
5,50%	18,2	20,0	22,2	25,0	28,6	33,3	40,0	50,0	66,7	100,0	200,0		
6,00%	16,7	18,2	20,0	22,2	25,0	28,6	33,3	40,0	50,0	66,7	100,0	200,0	

Table 7: Price-earnings ratio under different assumptions on the growth rate of dividends and the level of the interest rate.

Table 7 shows by which number the current dividend, D_{t+1}, has to be multiplied to arrive at the share price, P_t^S. As we can see, the price-earnings ratio, P_t^S/D_{t+1}, is very sensitive to the future growth of dividends and the future level of the interest rate.

7 Bond prices

The price at which a bond is issued (nominal value, face value or par value) is equal to its value at maturity:

$$P_0^B = P_T^B. \tag{37}$$

We make use of the no-arbitrage condition for the bond:

$$R_t^B = R_t. \tag{38}$$

Also note that if x, y and z are small:

$$\begin{aligned}
&1 + z = \frac{1+x}{1+y} && (39)\\
\Leftrightarrow\quad &\ln(1+z) = \ln(1+x) - \ln(1+y) && (40)\\
\Leftrightarrow\quad &z \approx x - y && (41)\\
\Leftrightarrow\quad &\frac{1+x}{1+y} \approx 1 + x - y. && (42)
\end{aligned}$$

Let C_t be the coupon rate of the bond in period t.

Now we can use the formula for the return on a bond in the final period, R_T^B, to compute the approximate price of the bond at dates $T-1$, $T-2$ etc.:

$$R_T^B = \frac{P_T^B - P_{T-1}^B + C_T P_0^B}{P_{T-1}^B} \tag{43}$$

$$\begin{aligned}
\Leftrightarrow\quad P_{T-1}^B &= \frac{P_T^B + C_T P_0^B}{1 + R_T}\\
&= \frac{1 + C_T}{1 + R_T} \times P_T^B\\
&\approx (1 + C_T - R_T) \times P_T^B,
\end{aligned} \tag{44}$$

$$\begin{aligned}
P_{T-2}^B &= \frac{P_{T-1}^B + C_{T-1} P_0^B}{1 + R_{T-1}}\\
&= \frac{P_{T-1}^B + C_{T-1} P_T^B}{1 + R_{T-1}}\\
&\approx \frac{(1 + C_T - R_T + C_{T-1})}{1 + R_{T-1}} \times P_T^B\\
&\approx (1 + C_T - R_T + C_{T-1} - R_{T-1}) \times P_T^B,
\end{aligned} \tag{45}$$

$$P_{T-h}^B \approx \left(1 + \sum_{i=0}^{h-1} (C_{T-i} - R_{T-i})\right) \times P_T^B. \tag{46}$$

8 Optimization

8.1 Optimization with equality constraints

8.1.1

$$\max f(x_1, x_2), \tag{47}$$

subject to

$$g(x_1, x_2) = b. \tag{48}$$

$$\max f(x_1, x_2, g(x_1, x_2)), \tag{49}$$

subject to

$$g(x_1, x_2) = b. \tag{50}$$

First-order conditions:

$$\begin{aligned} \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_1'(x_1, x_2) = 0, \\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_2'(x_1, x_2) = 0, \end{aligned} \tag{51}$$

where the chain rule is used.

Set $\lambda = f_3'(x_1, x_2, b)$, where the derivative is evaluated with x_1 and x_2 taking on the optimal values.

Then, since $g(x_1, x_2) = b$, we can rewrite the first-order conditions as:

$$\begin{aligned} \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2) + \lambda g_1'(x_1, x_2) = 0, \\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2) + \lambda g_2'(x_1, x_2) = 0, \end{aligned} \tag{52}$$

where the functions f_1 and f_2 are written again with two arguments for simplicity.

These conditions have to be solved together with the constraint:

$$g(x_1, x_2) = b. \tag{53}$$

This gives three equations with three unknowns (x_1, x_2 and λ).

8.2 Optimization with inequality constraints

8.2.1

$$\max f(x_1, x_2), \tag{54}$$

subject to

$$g(x_1, x_2) \leq b. \tag{55}$$

$$\max f(x_1, x_2, g(x_1, x_2)), \tag{56}$$

subject to

$$g(x_1, x_2) \leq b. \tag{57}$$

First-order conditions:

$$\begin{aligned}\frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_1'(x_1, x_2) = 0,\\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_2'(x_1, x_2) = 0,\end{aligned} \tag{58}$$

where the chain rule is used.

Set $\lambda = f_3'(x_1, x_2, b)$, where the derivative is evaluated with x_1 and x_2 taking on the optimal values.

Note that it is necessary that $\lambda = f_3'(x_1, x_2, g(x_1, x_2)) = f_3'(x_1, x_2, b) \geq 0$ (interior or corner solution). In this case, $g(x_1, x_2) = b$.

If no corner solution: $\lambda = f_3'(x_1, x_2, b) = 0$

If $g(x_1, x_2) < b$, the constraint is not binding. This means that the optimum must be the same no matter whether we optimize the functions f_1 and f_2 with two or with three arguments.

Hence:

$$\begin{aligned}\frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_1'(x_1, x_2) = 0,\\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2, g(x_1, x_2)) + f_3'(x_1, x_2, g(x_1, x_2))g_2'(x_1, x_2) = 0,\end{aligned} \tag{59}$$

Hence we can rewrite the first-order conditions as:

$$\begin{aligned}\frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_1} &= f_1'(x_1, x_2) + \lambda g_1'(x_1, x_2) = 0,\\ \frac{\partial f(x_1, x_2, g(x_1, x_2))}{\partial x_2} &= f_2'(x_1, x_2) + \lambda g_2'(x_1, x_2) = 0,\end{aligned} \tag{60}$$

where the functions f_1 and f_2 are written again with two arguments for simplicity.

These conditions have to be solved together with the constraint:

$$g(x_1, x_2) = b. \tag{61}$$

This gives three equations with three unknowns (x_1, x_2 and λ).

8.2.2 Lagrange method

$$\max f(x_1, \ldots, x_n), \tag{62}$$

subject to

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= b_1, \\ &\cdots \\ g_m(x_1, \ldots, x_n) &= b_m \end{aligned} \tag{63}$$

We assume that $m < n$.

By setting $\mathbf{x} = (x_1, \ldots, x_n)'$, the problem can be written more compactly as follows:

$$\max f(\mathbf{x}), \tag{64}$$

subject to

$$g_j(\mathbf{x}) = b_1, \quad j = 1, \ldots, m. \tag{65}$$

$$\tag{66}$$

To solve this problem, we set up the Lagrange function, or Lagrangian:

$$\mathcal{L}(\mathbf{x}) = f(\mathbf{x}) - \lambda_1 g_1(\mathbf{x}) - \ldots - \lambda_m g_m(\mathbf{x}), \tag{67}$$

where $\lambda_1, \ldots, \lambda_m$ are called Lagrange multipliers.

The first-order conditions are then:

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial x_i} = \frac{\partial f(\mathbf{x})}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0, \quad i = 1, \ldots, n. \tag{68}$$

We can now find the solution to the optimization problem in equations (62) and (63) by solving simultaneously the n first-order conditions in equation (68) and the m constraints in equation set (63) for the $n + m$ variables $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_m$.

Indeed, it can be shown that if we can find Lagrange multipliers $\lambda_1, \ldots, \lambda_m$ and a vector $\mathbf{x}^*$ that satisfy all of the first-order conditions and constraints and if the Lagrangian $\mathcal{L}(\mathbf{x})$ is concave in $\mathbf{x}$, then $\mathbf{x}^*$ solves the optimization problem in equations (62) and (63) (see, for example, **?**).

8.2.3 Examples

Example 1 Consider the following optimization problem:

$$\max_x -x^2 + 2, \tag{69}$$

subject to

$$x = b. \tag{70}$$

The Lagrangian is:

$$\mathcal{L}(x) = -x^2 + 2 - \lambda x = 0. \tag{71}$$

The first-order condition is:

$$\frac{d\mathcal{L}(x)}{dx} = -2x - \lambda = 0. \tag{72}$$

Hence:

$$x^* = b, \tag{73}$$
$$\lambda = -2b. \tag{74}$$

Note that the Lagrange multiplier $\lambda = \lambda(b)$ is the rate at which the optimal value of the objective function changes with respect to changes in the constant b:

$$\lambda(b) = \frac{df^*(b)}{db} = \frac{df(x^*(b))}{db} = \frac{-(x^*)^2 + 2}{dx^*} \times \frac{dx^*(b)}{db} = -2x^*. \tag{75}$$

In this simple example, x^* always equals b. Hence $f^*(b) = -b^2 + 2$ and $\lambda(b) = -2b$. For example, when $b = -1$, $f^*(b)$ rises at rate 2 with respect to b. When $b = 0$, $f^*(b) = -b^2 + 2$ stays constant when b changes marginally. And when $b = 1$, $f^*(b)$ falls at rate -2 with respect to b.

To simplify the notation, from now on we will omit the asterisk for solutions (for example, x instead of x^*).

Example 2 Things become more interesting when there are two or more variables. Consider the following optimization problem with two variables:

$$\max_{C_1, C_2} \ln(C_1) + \ln(C_2), \tag{76}$$

subject to

$$C_1 + C_2 = Y. \tag{77}$$

Note that it is possible to transform this problem into a problem of just one variable:

$$\max_{C_1, C_2} \ln(C_1) + \ln(Y - C_1). \tag{78}$$

The first-order condition is:

$$\frac{1}{C_1} - \frac{1}{Y - C_1} = 0. \tag{79}$$

Hence the solution is:

$$C_1 = C_2 = \frac{1}{2}Y. \tag{80}$$

The Lagrangian yields the same result, but provides us also with the shadow value of the constraint. The Lagrangian is:

$$\mathcal{L}(C_1, C_2) = \ln(C_1) + \ln(C_2) - \lambda(C_1 + C_2). \tag{81}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}(C_1, C_2)}{\partial C_1} = \frac{1}{C_1} - \lambda = 0, \tag{82}$$

$$\frac{\partial \mathcal{L}(C_1, C_2)}{\partial C_2} = \frac{1}{C_2} - \lambda = 0. \tag{83}$$

Again the solution is:

$$C_1 = C_2 = \frac{1}{2}Y, \tag{84}$$

but now we also find out that the marginal benefit of relaxing the constraint is:

$$\lambda = \frac{2}{Y}. \tag{85}$$