

# **International macroeconomics (advanced level)**

## **Lecture note additions**

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## Miscellaneous notes

### 1 The government's budget constraint and Ricardian equivalence

#### 1.1 Model with exogenous income

The maximization problem for the representative consumer is:

$$\max_{C_1} u(C_1) + \beta u(C_2), \quad (1)$$

where

$$C_2 = (1 + r)(Y_1 - T_1 - C_1) + Y_2 - T_2. \quad (2)$$

The first-order condition leads to the Euler equation:

$$u'(C_1) = \beta(1 + r)u'(C_2). \quad (3)$$

Assuming logarithmic utility, we get:

$$C_2 = \beta(1 + r)C_1. \quad (4)$$

Additional assumptions:

$$T_2 = (1 + r)(G_1 - T_1) + G_2, \quad (5)$$

$$S_1^p = Y_1 - T_1 - C_1. \quad (6)$$

The model consists of the four equations (2), (4), (5) and (6). Correspondingly, there are four endogenous variables:  $C_1$ ,  $C_2$ ,  $T_2$  and  $S_1^p$ . Solving for the endogenous variables, we get:

$$C_1 = \frac{1}{1 + \beta} \left( Y_1 - G_1 + \frac{Y_2 - G_2}{1 + r} \right), \quad (7)$$

$$C_2 = \frac{\beta}{1 + \beta} (1 + r) \left( Y_1 - G_1 + \frac{Y_2 - G_2}{1 + r} \right), \quad (8)$$

$$T_2 = (1 + r)(G_1 - T_1) + G_2, \quad (9)$$

$$S_1^p = Y_1 - T_1 - \frac{1}{1 + \beta} \left( Y_1 - G_1 + \frac{Y_2 - G_2}{1 + r} \right). \quad (10)$$

### 1.1.1 Result 1: Ricardian equivalence

Lowering  $T_1$  by 1 unit implies a rise of  $T_2$  by  $1 + r$  units. Hence  $C_1$  and  $C_2$  are not affected by the way taxes are distributed over the two periods. This is called Ricardian equivalence. A fall of  $T_1$  is matched by a corresponding rise in savings,  $S_1^p$ .

### 1.1.2 Result 2: Fiscal policy

A rise of  $G_1$  by 1 unit implies a fall (!) of  $C_1$  by  $1/(1 + \beta)$  units and of  $C_2$  by  $\beta(1 + r)/(1 + \beta)$  units. This is because higher current government spending,  $G_1$ , implies a rise in future taxation,  $T_2$ .

Similarly, a rise of  $G_2$  by 1 unit leads to a fall of  $C_1$  by  $1/[(1 + \beta)(1 + r)]$  units and of  $C_2$  by  $\beta/(1 + \beta)$  units.

## 1.2 Model with endogenous income

Suppose we add the following equation to the model given by the equations (2), (4), (5) and (6):

$$Y_1 = C_1 + G_1. \quad (11)$$

The model has now five equations. The five endogenous variables are:  $C_1$ ,  $C_2$ ,  $T_2$ ,  $Y_1$  and  $S_1^p$ . The solution is as follows:

$$C_1 = \frac{Y_2 - G_2}{\beta(1+r)}, \quad (12)$$

$$C_2 = Y_2 - G_2, \quad (13)$$

$$T_2 = (1+r)(G_1 - T_1) + G_2, \quad (14)$$

$$Y_1 = \frac{Y_2 - G_2}{\beta(1+r)} + G_1, \quad (15)$$

$$S_1^p = Y_1 - T_1 - \frac{Y_2 - G_2}{\beta(1+r)}. \quad (16)$$

### 1.2.1 Result 1: Ricardian equivalence

Ricardian equivalence still holds. Lowering  $T_1$  by 1 unit implies a rise of  $T_2$  by  $1+r$  units. Hence  $C_1$  and  $C_2$  are not affected by the way taxes are distributed over the two periods.

### 1.2.2 Result 2: Fiscal policy

Fiscal policy in the current period,  $G_1$ , leaves consumption in both periods unchanged. A rise in  $G_1$  by 1 unit raises  $Y_1$  by 1 unit (expansionary fiscal policy) and  $T_2$  by  $1+r$  units, leaving the present discounted value of the consumer's current and future disposable incomes unchanged.

A rise in future government spending,  $G_2$ , has a negative impact on consumption in both periods. The consumer knows that higher government spending in the second period has to be financed by higher future taxes, so he or she chooses to reduce consumption accordingly.



### 1.2.3 Result 3: Accelerator effect

Note that the effect of  $Y_2$  and  $G_2$  on present consumption,  $C_1$ , and income,  $Y_1$ , is now much stronger than in the previous model. The corresponding derivative of  $C_1$  with respect to  $G_2$  is now  $-1/[\beta(1+r)]$ , whereas before it was  $-1/[(1+\beta)(1+r)]$ . This is because there is now a feedback of current consumption on current income. A rise of, say,  $G_2$  by 1 unit makes  $C_1$  fall by  $1/[(1+\beta)(1+r)]$  units, this leads to fall of  $Y_1$  by the same amount, another fall of  $C_1$  by  $1/[(1+\beta)^2(1+r)]$  units, a corresponding fall of  $Y_1$  etc. Hence the total fall of  $C_1$  (and hence  $Y_1$ ) is:

$$\begin{aligned}\Delta C_1 &= -\frac{1}{1+\beta} \left( 1 + \frac{1}{1+\beta} + \frac{1}{(1+\beta)^2} + \frac{1}{(1+\beta)^3} + \dots \right) \frac{1}{1+r} \Delta G_2 \\ &= -\frac{1}{\beta} \frac{1}{1+r} \Delta G_2.\end{aligned}\tag{17}$$

Note that we have applied here the formula for the infinite geometric series. Provided  $|x| < 1$ ,

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}.\tag{18}$$

## 2 Digression on growth rates

### 2.1 Growth accounting

We can derive the relationships between growth rates by direct calculation in both discrete and continuous time.

In the continuous-time case, we may alternatively use "log-differentiation", that is:

- first take the logarithm of a variable and
- then differentiate the resulting logarithm with respect to time.

This produces exact growth rates since:

$$\frac{d \log(z_t)}{dt} = \frac{1}{z_t} \times \frac{dz_t}{dt} = \frac{\dot{z}_t}{z_t} = \hat{z}_t, \quad (19)$$

where the dot above  $z_t$  indicates the derivative of that variable with respect to time and where the hat above  $z_t$  indicates its percentage change.

In the discrete-time case, we may alternatively use "log-differencing", that is:

- first take the logarithm of a variable and
- then apply the difference operator.

This produces approximate growth rates since:

$$\Delta \log(z_{t+1}) = \log\left(\frac{z_{t+1}}{z_t}\right) = \log\left(1 + \frac{z_{t+1} - z_t}{z_t}\right) \approx \frac{z_{t+1} - z_t}{z_t} = \hat{z}_t, \quad (20)$$

where the approximation is good provided  $(z_{t+1} - z_t)/z_t$  is small.

### 2.1.1 Example 1: Summation of variables

Summation of variables:

$$z = x + y. \quad (21)$$

Direct calculation in discrete time:

$$\hat{z}_t = \frac{\Delta z_{t+1}}{z_t} = \frac{\Delta x_{t+1} + \Delta y_{t+1}}{z_t} = \frac{x_t \Delta x_{t+1}}{z_t x_t} + \frac{y_t \Delta y_{t+1}}{z_t y_t} = \frac{x_t}{z_t} \hat{x}_t + \frac{y_t}{z_t} \hat{y}_t. \quad (22)$$

Log-differencing:

$$\hat{z}_t \approx \Delta \log(z_{t+1}) = \Delta \log(x_{t+1} + y_{t+1}) \approx \frac{x_{t+1} + y_{t+1} - (x_t + y_t)}{z_t} = \frac{x_t}{z_t} \hat{x}_t + \frac{y_t}{z_t} \hat{y}_t. \quad (23)$$

Direct calculation in continuous time:

$$\hat{z} = \frac{\dot{z}}{z} = \frac{\dot{x} + \dot{y}}{x + y} = \frac{x \dot{x}}{z x} + \frac{y \dot{y}}{z y} = \frac{x}{z} \hat{x} + \frac{y}{z} \hat{y}. \quad (24)$$

Log-differentiation:

$$\hat{z} = \frac{d \log(z)}{dt} = \frac{d \log(x + y)}{dt} = \frac{\dot{x} + \dot{y}}{x + y} = \frac{x \dot{x}}{z x} + \frac{y \dot{y}}{z y} = \frac{x}{z} \hat{x} + \frac{y}{z} \hat{y}. \quad (25)$$

## 2.1.2 Example 2: Multiplication of variables

Multiplication of variables:

$$z = xy. \tag{26}$$

Direct calculation in discrete time:

$$\begin{aligned} \hat{z}_t &= \frac{\Delta z_{t+1}}{z_t} = \frac{\Delta(x_{t+1}y_{t+1})}{x_t y_t} = \frac{[x_t + \Delta x_{t+1}][y_t + \Delta y_{t+1}] - x_t y_t}{x_t y_t} \\ &= \hat{x}_t + \hat{y}_t + \hat{x}_t \hat{y}_t \approx \hat{x}_t + \hat{y}_t. \end{aligned} \tag{27}$$

Log-differencing:

$$\hat{z}_t \approx \Delta \log(z_{t+1}) = \Delta \log(x_{t+1}y_{t+1}) = \Delta \log(x_{t+1}) + \Delta \log(y_{t+1}) \approx \hat{x}_t + \hat{y}_t. \tag{28}$$

Direct calculation in continuous time:

$$\hat{z} = \frac{\dot{z}}{z} = \frac{\dot{x}y + x\dot{y}}{xy} = \hat{x} + \hat{y}. \tag{29}$$

Log-differentiation:

$$\hat{z} = \frac{d \log(z)}{dt} = \frac{d[\log(x) + \log(y)]}{dt} = \frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \hat{x} + \hat{y}. \tag{30}$$

### 2.1.3 Example 3: Division of variables

Division of variables:

$$z = \frac{x}{y}. \quad (31)$$

Direct calculation in discrete time:

$$\hat{z}_t = \frac{\Delta z_{t+1}}{z_t} = \frac{\frac{x_{t+1} + \Delta x_{t+1}}{y_{t+1} + \Delta y_{t+1}} - \frac{x_t}{y_t}}{\frac{x_t}{y_t}} = \frac{1 + \hat{x}_t - (1 + \hat{y}_t)}{1 + \hat{y}_t} = \frac{\hat{x}_t - \hat{y}_t}{1 + \hat{y}_t} \approx \hat{x}_t - \hat{y}_t. \quad (32)$$

Log-differencing:

$$\hat{z}_t \approx \Delta \log(z_{t+1}) = \Delta \log(x_{t+1}/y_{t+1}) = \Delta \log(x_{t+1}) - \Delta \log(y_{t+1}) \approx \hat{x}_t - \hat{y}_t. \quad (33)$$

Direct calculation in continuous time:

$$\hat{z} = \frac{\dot{z}}{z} = \frac{\frac{\dot{x}y - x\dot{y}}{y^2}}{\frac{x}{y}} = \frac{\dot{x}y - x\dot{y}}{xy} = \hat{x} - \hat{y}. \quad (34)$$

Log-differentiation:

$$\hat{z} = \frac{d \log(z)}{dt} = \frac{d[\log(x) - \log(y)]}{dt} = \frac{\dot{x}}{x} - \frac{\dot{y}}{y} = \hat{x} - \hat{y}. \quad (35)$$

### 2.1.4 Example 4: Multiplying a variable with a constant

Multiplying a variable with a constant:

$$z = ax. \tag{36}$$

Direct calculation in discrete time:

$$\hat{z}_t = \frac{\Delta z_{t+1}}{z_t} = \frac{ax_{t+1} - ax_t}{ax_t} = \frac{x_{t+1} - x_t}{x_t} = \hat{x}_t. \tag{37}$$

Log-differencing:

$$\begin{aligned} \hat{z}_t &\approx \Delta \log(z_{t+1}) = \Delta \log(ax_{t+1}) \\ &= \log(a) + \log(x_{t+1}) - [\log(a) + \log(x_t)] = \Delta \log(x_{t+1}) \approx \hat{x}_t. \end{aligned} \tag{38}$$

Direct calculation in continuous time:

$$\hat{z} = \frac{\dot{z}}{z} = \frac{a\dot{x}}{ax} = \frac{\dot{x}}{x} = \hat{x}. \quad (39)$$

Log-differentiation:

$$\hat{z} = \frac{d \log(z)}{dt} = \frac{d \log(ax)}{dt} = \frac{d \log(a)}{dt} + \frac{d \log(x)}{dt} = \frac{d \log(x)}{dt} = \hat{x}. \quad (40)$$



## 2.1.5 Example 5: Taking the power of a variable

Taking the power of a variable:

$$z = x^a. \quad (41)$$

Direct calculation in discrete time:

$$\hat{z}_t = \frac{\Delta z_{t+1}}{z_t} = \frac{x_{t+1}^a - x_t^a}{x_t^a} \approx \frac{x_t^a + ax_t^{a-1}\Delta x_{t+1} + \frac{1}{2}a(a-1)x_t^{a-2}(\Delta x_{t+1})^2 - x_t^a}{x_t^a} \approx a\hat{x}_t, \quad (42)$$

provided  $\Delta x_{t+1}$  is small, where  $x_{t+1}^a$  has been approximated using a second-order Taylor series expansion around  $x_t$ .

Log-differencing:

$$\hat{z}_t \approx \Delta \log(z_{t+1}) = \Delta \log(x_{t+1}^a) = a\Delta \log(x_{t+1}) \approx a\hat{x}_t. \quad (43)$$

Direct calculation in continuous time:

$$\hat{z} = \frac{\dot{z}}{z} = \frac{ax^{a-1}\dot{x}}{x^a} = a\hat{x}. \quad (44)$$

Log-differentiation:

$$\hat{z} = \frac{d \log(z)}{dt} = \frac{d \log(x^a)}{dt} = a \frac{d \log(x)}{dt} = a\hat{x}. \quad (45)$$

### 3 Elasticity of intertemporal substitution

An asset that is continuously reinvested with a net return of  $r$  has the following value  $V$ :

$$V_t = e^{rt}. \tag{46}$$

The net return during an infinitesimally small period  $dt$  is  $r$  since by L'Hôpital's rule:

$$\frac{dV_t}{V_t} = \lim_{dt \rightarrow 0} \frac{e^{rdt} - 1}{dt} = \lim_{dt \rightarrow 0} \frac{r e^{rdt}}{1} = r. \tag{47}$$

The elasticity of intertemporal substitution is defined as:

$$\sigma = \frac{d \frac{dC}{C}}{d \frac{dV_t}{V_t}} = \frac{d \frac{dC}{C}}{dr}. \tag{48}$$

It turns out that the elasticity of intertemporal substitution,  $\sigma$ , is equal to the inverse of the parameter of relative risk aversion,  $\gamma$ . To see this, one can combine the above definition of the elasticity of intertem-

poral substitution with the Euler equation that results from optimizing consumption over time. The Euler equation yields:

$$u'(C) = \beta(1+r)u'(C+dC) \quad (49)$$

$$\Rightarrow r \approx \ln(1+r) = -\ln(\beta) - \ln\left(\frac{u'(C+dC)}{u'(C)}\right) = -\ln(\beta) - \frac{du'(C)}{u'(C)} \quad (50)$$

$$\Rightarrow dr = -d \ln\left(\frac{u'(C+dC)}{u'(C)}\right) = -d \frac{du'(C)}{u'(C)}. \quad (51)$$

Now substitute the return of the asset,  $r$ , into the definition of the elasticity of intertemporal substitution,  $\sigma$ :

$$\sigma = \frac{d\frac{dC}{C}}{dr} = -\frac{d\frac{dC}{C}}{d\frac{du'(C)}{u'(C)}} = -\frac{u'(C)}{Cu''(C)} = \frac{1}{\gamma}. \quad (52)$$